THE UNIVERSITY OF GLASGOW

DEPARTMENT OF AERONAUTICS AND FLUID MECHANICS

STRESS ANALYSIS OF THE MICHELL CANTILEVER BEAM

by

A. Caldwell B. Sc.

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SUMMARY

The theory of layout of optimum two-dimensional frameworks is developed using directly the idea of compatibility of strain in the virtual deformation determined by Michell. Appropriate equilibrium equations are derived and solved for the special case of an equiangular spiral layout used to form a cantilever beam. It is concluded that the mathematical complexity which arises in dealing with this relatively simple layout raises doubts as to the value of seeking analytic solutions of the layout and equilibrium equations.

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1. Theory of Layout

In his original paper*, Michell derived the equations governing the layout lines of an optimum framework, containing both tension and compression members, by using the two conditions that the bars must be orthogonal before and after the virtual deformation, and this method of proof has been adopted by all subsequent writers on the subject. The bars must be orthogonal before the deformation since they must lie along the principal directions defined by the virtual strain tensor; and if the deformation is to be compatible, the bars necessarily remain orthogonal.

In this presentation the condition of compatibility of strain is used directly to derive the equations governing the layout lines. In this way, it is possible to eliminate the physical idea of strain. The conditions to be satisfied may be stated as:-

1. The bars must lie along the principal directions defined by a second order tensor A_{ij}.

2. A_{ij} must form the symmetric part of $B_{i,j}$ where B_i is a first order tensor (vector) whose components are continuous, and must be such that when referred to principal axes,

$$A_{ij} = B_{i,j} = k \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad k = \text{constant.}$$

(attention is confined to plane systems)

These conditions are very contrived, and their relation to the virtual strain conditions is obvious ($A_{ij} = e_{ij}$ = strain tensor: $B_i = u_i$ = displacement vector: $k = \epsilon$ = magnitude of strain: condition 2 is merely

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* Reference (4)

the compatibility of strain condition. There is little to be gained in developing a mathematical argument leading to Michell's Theorem which is divorced from the physical context of compatibility of strain (although the Appendix shows that some short cuts in the derivation of the equations of the layout lines may be taken) and therefore in what follows the idea of virtual strain is retained.

2. Derivation of Equations of Layout Lines

Further mathematical development is best achieved by means of the tensor calculus (reference 1). The notation and terminology are standard. The layout lines are used as a system of orthogonal curvilinear coordinates: the fundamental tensor, $g_{\alpha\beta}$, which is necessarily diagonal for an orthogonal system, is given by:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

Let $g_{\alpha\beta} = \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix}$ i.e. $ds^2 = A^2\, d\alpha^2 + B^2\, d\beta^2$

The subscript or superscript 1 or α will denote the α-direction and the subscript or superscript 2 or β will denote the β-direction.

The conjugate of the tensor $g_{\alpha\beta}$ is

$$g^{\alpha\beta} = \begin{bmatrix} \frac{1}{A^2} & 0 \\ 0 & \frac{1}{B^2} \end{bmatrix}$$

Christoffel Symbols of the First Kind

$$\left[11,\,2\right] = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}\right) = -A\,\frac{\partial A}{\partial \beta}$$

$$\left[12,\,1\right] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^1}\right) = A\,\frac{\partial A}{\partial \beta}$$

$$\left[21,\,1\right] = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^2} - \frac{\partial g_{21}}{\partial x^1}\right) = A\,\frac{\partial A}{\partial \beta}$$

$$\left[11,\,1\right] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\right) = A\,\frac{\partial A}{\partial \alpha}$$

$$\left[22,\,2\right] = \frac{1}{2}\left(\frac{\partial g_{22}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^2}\right) = B\,\frac{\partial B}{\partial \beta}$$

$$\left[22,\,1\right] = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right) = -B\,\frac{\partial B}{\partial \alpha}$$

$$\left[21,\,2\right] = \frac{1}{2}\left(\frac{\partial g_{22}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{21}}{\partial x^2}\right) = B\,\frac{\partial B}{\partial \alpha}$$

$$\left[12,\,2\right] = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^2}\right) = B\,\frac{\partial B}{\partial \alpha}$$

Christoffel Symbols of the Second Kind

$$\left\{ {1 \atop 12} \right\} = g^{11} \left[12, 1\right] + g^{12} \left[12, 2\right] = \frac{1}{A} \frac{\partial A}{\partial \beta}$$

$$\left\{ {1 \atop 21} \right\} = g^{11} \left[21, 1\right] + g^{12} \left[21, 2\right] = \frac{1}{A} \frac{\partial A}{\partial \beta}$$

$$\left\{ {2 \atop 11} \right\} = g^{21} \left[11, 1\right] + g^{22} \left[11, 2\right] = \frac{A}{B^2} \frac{\partial A}{\partial \beta}$$

$$\left\{ {1 \atop 11} \right\} = g^{11} \left[11, 1\right] + g^{12} \left[11, 2\right] = \frac{1}{A} \frac{\partial A}{\partial \alpha}$$

$$\left\{ {2 \atop 22} \right\} = g^{21} \left[22, 1\right] + g^{22} \left[22, 2\right] = \frac{1}{B} \frac{\partial B}{\partial \beta}$$

$$\left\{ {2 \atop 21} \right\} = g^{21} \left[21, 1\right] + g^{22} \left[21, 2\right] = \frac{1}{B} \frac{\partial B}{\partial \alpha}$$

$$\left\{ {2 \atop 12} \right\} = g^{21} \left[12, 1\right] + g^{22} \left[12, 2\right] = \frac{1}{B} \frac{\partial B}{\partial \alpha}$$

$$\left\{ {1 \atop 22} \right\} = g^{11} \left[22, 1\right] + g^{12} \left[22, 2\right] = -\frac{B}{A^2} \frac{\partial B}{\partial \alpha}$$

Strain Tensor

Strain Tensor $\quad e_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$

Let $\quad (u_\alpha, u_\beta) = (u, v) =$ displacement vector.

$$e_{\alpha\alpha} = \frac{\partial u}{\partial \alpha} - \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} u - \begin{Bmatrix} 2 \\ 11 \end{Bmatrix} v$$

$$= \frac{\partial u}{\partial \alpha} - \frac{1}{A}\frac{\partial A}{\partial \alpha} u + \frac{A}{B^2}\frac{\partial A}{\partial \beta} v$$

$$e_{\beta\beta} = \frac{\partial v}{\partial \beta} - \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} u - \begin{Bmatrix} 2 \\ 22 \end{Bmatrix} v$$

$$= \frac{\partial v}{\partial \beta} + \frac{B}{A^2}\frac{\partial B}{\partial \alpha} u - \frac{1}{B}\frac{\partial B}{\partial \beta} v$$

$$e_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) - \frac{1}{2}\left[\begin{Bmatrix} 1 \\ 12 \end{Bmatrix} u - \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} v - \begin{Bmatrix} 1 \\ 21 \end{Bmatrix} u - \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} v\right]$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) - \frac{1}{A}\frac{\partial A}{\partial \beta} u - \frac{1}{B}\frac{\partial B}{\partial \alpha} v$$

Physical Components of Strain

The components of a tensor, when transformed from a cartesian system to another reference system may lose their physical significance and dimensionality. To obtain the physical components of the strain tensor it is necessary to transform the quantities $e_{\alpha\beta} l^\alpha m^\beta$ where l^α and m^β are the unit contravariant vectors in the α- and β-directions respectively, i.e.

$$l^{\alpha} = \frac{\delta_1^{\ \alpha}}{\sqrt{g_{11}}} \quad , \quad m^{\beta} \quad , \quad \frac{\delta_2^{\ \beta}}{\sqrt{g_{22}}}$$

Therefore, denoting the physical strains by $\epsilon_{\alpha\beta}$:

$$\epsilon_{\alpha\alpha} = \frac{1}{g_{11}} e_{\alpha\alpha} = \frac{1}{A^2}\frac{\partial u}{\partial \alpha} - \frac{u}{A^3}\frac{\partial A}{\partial \alpha} + \frac{v}{AB^2}\frac{\partial A}{\partial \beta}$$

$$\epsilon_{\beta\beta} = \frac{1}{g_{22}} e_{\beta\beta} = \frac{1}{B^2}\frac{\partial v}{\partial \beta} - \frac{v}{B^3}\frac{\partial B}{\partial \beta} + \frac{u}{A^2B}\frac{\partial B}{\partial \alpha}$$

$$\epsilon_{\alpha\beta} = \frac{1}{\sqrt{g_{11}g_{22}}} e_{\alpha\beta} = \frac{1}{2AB}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) - \frac{u}{A^2B}\frac{\partial A}{\partial \beta} - \frac{v}{AB^2}\frac{\partial B}{\partial \alpha}$$

The physical displacement vector (u', v') is given by

$$u' = \frac{u}{\sqrt{g_{11}}} = \frac{u}{A} \quad ; \quad v' = \frac{v}{\sqrt{g_{22}}} = \frac{v}{B}$$

Therefore finally:

$$\epsilon_{\alpha\alpha} = \frac{1}{A^2}\frac{\partial}{\partial \alpha}(u'A) - \frac{u'}{A^2}\frac{\partial A}{\partial \alpha} + \frac{v'}{AB}\frac{\partial A}{\partial \beta}$$

$$= \frac{1}{A}\frac{\partial u'}{\partial \alpha} + \frac{v'}{AB}\frac{\partial A}{\partial \beta}$$

$$\epsilon_{\beta\beta} = \frac{1}{B^2}\frac{\partial}{\partial\beta}(v'B) - \frac{v'}{B^2}\frac{\partial B}{\partial\beta} + \frac{u'}{AB}\frac{\partial B}{\partial\alpha}$$

$$= \frac{1}{B}\frac{\partial v'}{\partial\beta} + \frac{u'}{AB}\frac{\partial B}{\partial\alpha}$$

$$\epsilon_{\alpha\beta} = \frac{1}{2AB}\left(\frac{\partial}{\partial\beta}(u'A) + \frac{\partial}{\partial\alpha}(v'B)\right) - \frac{u'}{AB}\frac{\partial A}{\partial\beta} - \frac{v'}{AB}\frac{\partial B}{\partial\alpha}$$

$$= \frac{1}{2}\left[\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v'}{B}\right) + \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u'}{A}\right)\right]$$

Compatibility of Strain

The equations of compatibility of strain in tensor form are :

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$$

In a two dimensional system only one independent equation is formed. To form the derivatives $e_{ij,kl}$ it is first necessary to form the first order derivatives of the type $e_{ij,k}$.

In general:

$$e_{\alpha\beta},_{\gamma} = \frac{\partial e_{\alpha\beta}}{\partial x\gamma} - \begin{Bmatrix}\delta\\ \alpha\gamma\end{Bmatrix} e_{\delta\beta} - \begin{Bmatrix}\delta\\ \beta\gamma\end{Bmatrix} e_{\alpha\delta}$$

$$\therefore \quad e_{11,2} = \frac{\partial e_{\alpha\alpha}}{\partial\beta} - \begin{Bmatrix}1\\ 12\end{Bmatrix} e_{\alpha\alpha} - \begin{Bmatrix}2\\ 12\end{Bmatrix} e_{\alpha\beta} - \begin{Bmatrix}1\\ 12\end{Bmatrix} e_{\alpha\alpha} - \begin{Bmatrix}2\\ 12\end{Bmatrix} e_{\alpha\beta}$$

$$= \frac{\partial e_{\alpha\alpha}}{\partial\beta} - \frac{2}{A}\cdot\frac{\partial A}{\partial\beta} e_{\alpha\alpha} - \frac{2}{B}\frac{\partial B}{\partial\alpha} e_{\alpha\beta}$$

$$e_{11,1} = \frac{\partial e_{\alpha\alpha}}{\partial\alpha} - 2\begin{Bmatrix}1\\11\end{Bmatrix} e_{\alpha\alpha} - 2\begin{Bmatrix}2\\11\end{Bmatrix} e_{\alpha\beta}$$

$$= \frac{\partial e_{\alpha\alpha}}{\partial\alpha} - \frac{2}{A}\frac{\partial A}{\partial\alpha} e_{\alpha\alpha} + \frac{2A}{B^2}\frac{\partial A}{\partial\beta} e_{\alpha\beta}$$

$$e_{22,2} = \frac{\partial e_{\beta\beta}}{\partial\beta} - 2\begin{Bmatrix}1\\22\end{Bmatrix} e_{\alpha\beta} - 2\begin{Bmatrix}2\\22\end{Bmatrix} e_{\beta\beta}$$

$$= \frac{\partial e_{\beta\beta}}{\partial\beta} - \frac{2}{B}\frac{\partial B}{\partial\beta} e_{\beta\beta} + \frac{2B}{A^2}\frac{\partial B}{\partial\alpha} e_{\alpha\beta}$$

$$e_{22,1} = \frac{\partial e_{\beta\beta}}{\partial\alpha} - 2\begin{Bmatrix}1\\21\end{Bmatrix} e_{\alpha\beta} - 2\begin{Bmatrix}2\\21\end{Bmatrix} e_{\beta\beta}$$

$$= \frac{\partial e_{\beta\beta}}{\partial\alpha} - \frac{2}{A}\frac{\partial A}{\partial\beta} e_{\alpha\beta} - \frac{2}{B}\frac{\partial B}{\partial\alpha} e_{\beta\beta}$$

$$e_{21,1} = e_{12,1} = \frac{\partial e_{\alpha\beta}}{\partial\alpha} - \begin{Bmatrix}1\\21\end{Bmatrix} e_{\alpha\alpha} - \begin{Bmatrix}2\\21\end{Bmatrix} e_{\alpha\beta} - \begin{Bmatrix}1\\11\end{Bmatrix} e_{\alpha\beta} - \begin{Bmatrix}2\\11\end{Bmatrix} e_{\beta\beta}$$

$$= \frac{\partial e_{\alpha\beta}}{\partial\alpha} - \frac{1}{A}\frac{\partial A}{\partial\beta} e_{\alpha\alpha} + \frac{A}{B^2}\frac{\partial A}{\partial\beta} e_{\beta\beta} - \left(\frac{1}{B}\frac{\partial B}{\partial\alpha} + \frac{1}{A}\frac{\partial A}{\partial\alpha}\right) e_{\alpha\beta}$$

$$e_{21,2} = e_{12,2} = \frac{\partial e_{\alpha\beta}}{\partial\beta} - \begin{Bmatrix}1\\22\end{Bmatrix} e_{\alpha\alpha} - \begin{Bmatrix}2\\22\end{Bmatrix} e_{\alpha\beta} - \begin{Bmatrix}1\\12\end{Bmatrix} e_{\alpha\beta} - \begin{Bmatrix}2\\12\end{Bmatrix} e_{\beta\beta}$$

$$= \frac{\partial e_{\alpha\beta}}{\partial\beta} + \frac{B}{A^2}\frac{\partial B}{\partial\alpha} e_{\alpha\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha} e_{\beta\beta} - \left(\frac{1}{A}\frac{\partial A}{\partial\beta} + \frac{1}{B}\frac{\partial B}{\partial\beta}\right) e_{\alpha\beta}$$

$$e_{11,22} = \frac{\partial e_{11,2}}{\partial \beta} - \left\{ {1 \atop 12} \right\} e_{11,2} - \left\{ {2 \atop 12} \right\} e_{21,2} - \left\{ {1 \atop 12} \right\} e_{11,2}$$

$$- \left\{ {2 \atop 12} \right\} e_{12,2} - \left\{ {1 \atop 22} \right\} e_{11,1} - \left\{ {2 \atop 22} \right\} e_{11,2}$$

$$= \frac{\partial e_{11,2}}{\partial \beta} - \left(\frac{2}{A}\frac{\partial A}{\partial \beta} + \frac{1}{B}\frac{\partial B}{\partial \beta} \right) \left(\frac{\partial e_{\alpha\alpha}}{\partial \beta} - \frac{2}{A}\frac{\partial A}{\partial \beta} e_{\alpha\alpha} - \frac{2}{B}\frac{\partial B}{\partial \alpha} e_{\alpha\beta} \right)$$

$$- \frac{2}{B}\frac{\partial B}{\partial \alpha} \left(\frac{\partial e_{\alpha\beta}}{\partial \beta} + \frac{B}{A^2}\frac{\partial B}{\partial \alpha} e_{\alpha\alpha} - \frac{1}{B}\frac{\partial B}{\partial \alpha} e_{\beta\beta} - \left(\frac{1}{A}\frac{\partial A}{\partial \beta} + \frac{1}{B}\frac{\partial B}{\partial \beta} \right) e_{\alpha\beta} \right)$$

$$+ \frac{B}{A^2}\frac{\partial B}{\partial \alpha} \left(\frac{\partial e_{\alpha\alpha}}{\partial \alpha} - \frac{2}{A}\frac{\partial A}{\partial \alpha} e_{\alpha\alpha} + \frac{2A}{B^2}\frac{\partial A}{\partial \beta} e_{\alpha\beta} \right)$$

But $\varepsilon_{\alpha\beta} = 0$, $\varepsilon_{\alpha\alpha} = \varepsilon$, $\varepsilon_{\beta\beta} = -\varepsilon$

$\therefore$ $e_{\alpha\beta} = 0$, $e_{\alpha\alpha} = \varepsilon A^2$, $e_{\beta\beta} = -\varepsilon B^2$

$$\therefore e_{11,22} = \frac{\partial^2}{\partial \beta^2}(\varepsilon A^2) - \frac{\partial}{\partial \beta}\left(\frac{2}{A}\frac{\partial A}{\partial \beta} \varepsilon A^2 \right)$$

$$- \left(\frac{2}{A}\frac{\partial A}{\partial \beta} + \frac{1}{B}\frac{\partial B}{\partial \beta} \right) \left(2\varepsilon A \frac{\partial A}{\partial \beta} - 2\varepsilon A \frac{\partial A}{\partial \beta} \right)$$

$$- \left(\frac{2}{B}\frac{\partial B}{\partial \alpha} \right) \left(\varepsilon B \frac{\partial B}{\partial \alpha} + \varepsilon B \frac{\partial B}{\partial \alpha} \right)$$

$$+ \frac{B}{A^2}\frac{\partial B}{\partial \alpha} \left(2\varepsilon A \frac{\partial A}{\partial \alpha} - 2\varepsilon A \frac{\partial A}{\partial \alpha} \right)$$

$$= \frac{\partial^2}{\partial \beta^2}(\varepsilon A^2) - \frac{\partial^2}{\partial \beta^2}(\varepsilon A^2) - 4\varepsilon \left(\frac{\partial B}{\partial \alpha} \right)^2$$

$$= -4\varepsilon \left(\frac{\partial B}{\partial \alpha} \right)^2$$

$$e_{22,11} = \frac{\partial e_{22,1}}{\partial\alpha} - \left\{ {1 \atop 21} \right\} e_{12,1} - \left\{ {2 \atop 21} \right\} e_{22,1} - \left\{ {1 \atop 21} \right\} e_{21,1}$$

$$- \left\{ {2 \atop 21} \right\} e_{22,1} - \left\{ {1 \atop 11} \right\} e_{22,1} - \left\{ {2 \atop 11} \right\} e_{22,2}$$

$$= \frac{\partial e_{22,1}}{\partial\alpha} - \left(\frac{2}{B}\frac{\partial B}{\partial\alpha} + \frac{1}{A}\frac{\partial A}{\partial\alpha}\right)\left(\frac{\partial e_{\beta\beta}}{\partial\alpha} - \frac{2}{B}\frac{\partial B}{\partial\alpha} e_{\beta\beta} - \frac{2}{A}\frac{\partial A}{\partial\beta} e_{\alpha\beta}\right)$$

$$- \frac{2}{A}\frac{\partial A}{\partial\beta}\left(\frac{\partial e_{\alpha\beta}}{\partial\alpha} - \frac{1}{A}\frac{\partial A}{\partial\beta} e_{\alpha\alpha} + \frac{A}{B^2}\frac{\partial A}{\partial\beta} e_{\beta\beta} - \left(\frac{1}{B}\frac{\partial B}{\partial\alpha} + \frac{1}{A}\frac{\partial A}{\partial\alpha}\right) e_{\alpha\beta}\right)$$

$$+ \frac{A}{B^2}\frac{\partial A}{\partial\beta}\left(\frac{\partial e_{\beta\beta}}{\partial\beta} + \frac{2B}{A^2}\frac{\partial B}{\partial\alpha} e_{\alpha\beta} - \frac{2}{B}\frac{\partial B}{\partial\beta} e_{\beta\beta}\right)$$

$$= -\frac{\partial^2}{\partial\alpha^2}(\varepsilon B^2) + \frac{\partial}{\partial\alpha}\left(\frac{2}{B}\frac{\partial B}{\partial\alpha}\varepsilon B^2\right)$$

$$- \left(\frac{2}{B}\frac{\partial B}{\partial\alpha} + \frac{1}{A}\frac{\partial A}{\partial\alpha}\right)\left(-2\varepsilon B\frac{\partial B}{\partial\alpha} + 2\varepsilon B\frac{\partial B}{\partial\alpha}\right)$$

$$- \frac{2}{A}\frac{\partial A}{\partial\beta}\left(-A\varepsilon\frac{\partial A}{\partial\beta} - A\varepsilon\frac{\partial A}{\partial\beta}\right)$$

$$+ \frac{A}{B^2}\frac{\partial A}{\partial\beta}\left(-2\varepsilon B\frac{\partial B}{\partial\beta} + 2\varepsilon B\frac{\partial B}{\partial\beta}\right)$$

$$= -\frac{\partial^2}{\partial\alpha^2}(\varepsilon B^2) + \frac{\partial^2}{\partial\alpha^2}(\varepsilon B^2) + 4\varepsilon\left(\frac{\partial A}{\partial\beta}\right)^2$$

$$= 4\varepsilon\left(\frac{\partial A}{\partial\beta}\right)^2$$

$$e_{12,12} = \frac{\partial e_{12,1}}{\partial \beta} - \begin{Bmatrix}1\\12\end{Bmatrix} e_{12,1} - \begin{Bmatrix}2\\12\end{Bmatrix} e_{22,1} - \begin{Bmatrix}1\\22\end{Bmatrix} e_{11,1}$$

$$- \begin{Bmatrix}2\\22\end{Bmatrix} e_{12,1} - \begin{Bmatrix}1\\12\end{Bmatrix} e_{12,1} - \begin{Bmatrix}2\\12\end{Bmatrix} e_{12,2}$$

$$= \frac{\partial e_{12,1}}{\partial \beta} - \left(\frac{2}{A}\frac{\partial A}{\partial \beta} + \frac{1}{B}\frac{\partial B}{\partial \beta}\right)\left(\frac{\partial e_{\alpha\beta}}{\partial \alpha} - \frac{1}{A}\frac{\partial A}{\partial \beta} e_{\alpha\alpha} + \frac{A}{B^2}\frac{\partial A}{\partial \beta} e_{\beta\beta} - \left(\frac{1}{B}\frac{\partial B}{\partial \alpha} + \frac{1}{A}\frac{\partial A}{\partial \alpha}\right) e_{\alpha\beta}\right)$$

$$- \frac{1}{B}\frac{\partial B}{\partial \alpha}\left(\frac{\partial e_{\beta\beta}}{\partial \alpha} - \frac{2}{A}\frac{\partial A}{\partial \beta} e_{\alpha\beta} - \frac{2}{B}\frac{\partial B}{\partial \alpha} e_{\beta\beta}\right)$$

$$+ \frac{B}{A^2}\frac{\partial B}{\partial \alpha}\left(\frac{\partial e_{\alpha\alpha}}{\partial \alpha} - \frac{2}{A}\frac{\partial A}{\partial \alpha} e_{\alpha\alpha} + \frac{2A}{B^2}\frac{\partial A}{\partial \beta} e_{\alpha\beta}\right)$$

$$- \frac{1}{B}\frac{\partial B}{\partial \alpha}\left(\frac{\partial e_{\alpha\beta}}{\partial \beta} + \frac{B}{A^2}\frac{\partial B}{\partial \alpha} e_{\alpha\alpha} - \frac{1}{B}\frac{\partial B}{\partial \alpha} e_{\beta\beta} - \left(\frac{1}{A}\frac{\partial A}{\partial \beta} + \frac{1}{B}\frac{\partial B}{\partial \beta}\right) e_{\alpha\beta}\right)$$

$$= \frac{\partial}{\partial \beta}\left(-2A\varepsilon\frac{\partial A}{\partial \beta}\right) - \left(\frac{2}{A}\frac{\partial A}{\partial \beta} + \frac{1}{B}\frac{\partial B}{\partial \beta}\right)\left(-A\varepsilon\frac{\partial A}{\partial \beta} - A\varepsilon\frac{\partial A}{\partial \beta}\right)$$

$$- \frac{1}{B}\frac{\partial B}{\partial \alpha}\left(-2B\varepsilon\frac{\partial B}{\partial \alpha} + 2B\varepsilon\frac{\partial B}{\partial \alpha}\right)$$

$$+ \frac{B}{A^2}\frac{\partial B}{\partial \alpha}\left(2A\varepsilon\frac{\partial A}{\partial \alpha} - 2A\varepsilon\frac{\partial A}{\partial \alpha}\right)$$

$$- \frac{1}{B}\frac{\partial B}{\partial \alpha}\left(B\varepsilon\frac{\partial B}{\partial \alpha} + B\varepsilon\frac{\partial B}{\partial \alpha}\right)$$

$$= -2\varepsilon\left(\frac{\partial A}{\partial\beta}\right)^2 - 2A\varepsilon\frac{\partial^2 A}{\partial\beta^2} + 4\varepsilon\left(\frac{\partial A}{\partial\beta}\right)^2 + \frac{2A\varepsilon}{B}\left(\frac{\partial A}{\partial\beta}\right)\left(\frac{\partial B}{\partial\beta}\right) - 2\varepsilon\left(\frac{\partial B}{\partial\alpha}\right)^2$$

$$= 2\varepsilon\left(\frac{\partial A}{\partial\beta}\right)^2 - 2\varepsilon\left(\frac{\partial B}{\partial\alpha}\right)^2 - 2\varepsilon A\frac{\partial^2 A}{\partial\beta^2} + \frac{2A\varepsilon}{B}\left(\frac{\partial A}{\partial\beta}\right)\left(\frac{\partial B}{\partial\beta}\right)$$

Thus

$$e_{11,22} + e_{22,11} - 2e_{12,12} = 0$$

$$\Rightarrow -4\varepsilon\left(\frac{\partial B}{\partial\alpha}\right)^2 + 4\varepsilon\left(\frac{\partial A}{\partial\beta}\right)^2 - 4\varepsilon\left(\frac{\partial A}{\partial\beta}\right)^2 + 4\varepsilon\left(\frac{\partial B}{\partial\alpha}\right)^2 + 4\varepsilon A\frac{\partial^2 A}{\partial\beta^2} - \frac{4\varepsilon A}{B}\left(\frac{\partial A}{\partial\beta}\right)\left(\frac{\partial B}{\partial\beta}\right) = 0$$

$$\Rightarrow \frac{1}{B}\frac{\partial^2 A}{\partial\beta^2} - \frac{1}{B^2}\left(\frac{\partial A}{\partial\beta}\right)\left(\frac{\partial B}{\partial\beta}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = 0$$

This is not the only condition to be satisfied by the metric. The Euclidean Plane is a Flat Space and therefore the Riemann-Christoffel Tensor must be zero, i.e.

$$R^{l}_{jnp} = \left[\frac{\partial}{\partial x^n} \begin{Bmatrix} l \\ jp \end{Bmatrix} - \frac{\partial}{\partial x^p} \begin{Bmatrix} l \\ jn \end{Bmatrix} + \begin{Bmatrix} l \\ ns \end{Bmatrix} \begin{Bmatrix} s \\ jp \end{Bmatrix} - \begin{Bmatrix} l \\ ps \end{Bmatrix} \begin{Bmatrix} s \\ jn \end{Bmatrix} \right] = [0]$$

In a two dimensional space, this gives sixteen equations, fourteen of which are identically satisfied and two of which reduce to the same condition. Take $l = p = 1, \; n = j = 2$: then

$$\frac{\partial}{\partial\beta}\left(\frac{1}{A}\frac{\partial A}{\partial\beta}\right) - \frac{\partial}{\partial\alpha}\left(\frac{-B}{A^2}\frac{\partial B}{\partial\alpha}\right) + \left(\frac{1}{A}\frac{\partial A}{\partial\beta}\right)^2 - \left(\frac{B}{A^2}\frac{\partial B}{\partial\alpha}\right)\left(\frac{1}{B}\frac{\partial B}{\partial\alpha}\right)$$

$$- \left(\frac{1}{A}\frac{\partial A}{\partial\alpha}\right)\left(\frac{-B}{A^2}\frac{\partial B}{\partial\alpha}\right) - \left(\frac{1}{A}\frac{\partial A}{\partial\beta}\right)\left(\frac{1}{B}\frac{\partial B}{\partial\beta}\right) = 0$$

$$\Rightarrow \frac{1}{A}\frac{\partial^2 A}{\partial\beta^2} - \frac{1}{A^2}\left(\frac{\partial A}{\partial\beta}\right)^2 + \frac{B}{A^2}\frac{\partial^2 B}{\partial\alpha^2} + \frac{1}{A^2}\left(\frac{\partial B}{\partial\alpha}\right)^2 - \frac{2B}{A^3}\left(\frac{\partial B}{\partial\alpha}\right)\left(\frac{\partial A}{\partial\alpha}\right)$$

$$+ \frac{1}{A^2}\left(\frac{\partial A}{\partial\beta}\right)^2 - \frac{1}{A^2}\left(\frac{\partial B}{\partial\alpha}\right)^2 + \frac{B}{A^3}\left(\frac{\partial A}{\partial\alpha}\right)\left(\frac{\partial B}{\partial\alpha}\right) - \frac{1}{AB}\left(\frac{\partial A}{\partial\beta}\right)\left(\frac{\partial B}{\partial\beta}\right) = 0$$

$$\Rightarrow \frac{1}{B}\frac{\partial^2 A}{\partial\beta^2} - \frac{1}{B^2}\frac{\partial A}{\partial\beta}\frac{\partial B}{\partial\beta} + \frac{1}{A}\frac{\partial^2 B}{\partial x^2} - \frac{1}{A^2}\frac{\partial A}{\partial\alpha}\frac{\partial B}{\partial\alpha} = 0$$

$$\Rightarrow \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) + \frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) = 0$$

The equations to be satisfied by the layout lines are thus :

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) = 0$$

$$\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = 0 \qquad (2.1)$$

These equations may be reconciled with the ones obtained by Michell as follows.

Let ψ be the angle between a coordinate curve and some fixed direction: ψ is positive when rotation from the fixed direction to the positive direction of the curve is anti-clockwise. Then

$$\frac{\partial \psi}{\partial s}(\beta)^{*} = \frac{1}{B}\frac{\partial \psi}{\partial \beta}(\beta) = \sigma_{(\beta)} = \text{curvature of } \beta\text{-curves.}$$

(s= arc distance)

There is also the relation

$$\sigma_{(\beta)} n^{\beta} = \frac{\delta t^{\beta}}{\delta s} \quad (2.2) \quad \text{where} \quad t^{\beta} = \frac{\delta_2^{\,k}}{\sqrt{g_{22}}} = \text{unit tangent vector to } \beta\text{-curves,}$$

$$n^{\beta} = -\frac{\delta_1^{\,k}}{\sqrt{g_{11}}} = \text{unit normal vector to } \beta\text{-curves (see Fig.1),}$$

$\frac{\delta}{\delta s}$ denotes intrinsic derivative w.r.t. s., i.e.

$$\frac{\delta A^k}{\delta s} = \frac{dA^k}{ds} + \left\{ {k \atop rj} \right\} A^r \frac{dx^j}{ds}$$

where A^k is a vector.

---

* The brackets are introduced to avoid confusion with the tensor subscript.

$$\therefore \quad \frac{\delta t^{\beta}}{\delta s} = \left(\left[\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} \frac{1}{\sqrt{g_{22}}} \cdot \frac{1}{\sqrt{g_{22}}} \right], \left[-\frac{1}{2} g_{22}^{-\frac{3}{2}} \cdot \frac{1}{\sqrt{g_{22}}} \frac{\partial g_{22}}{\partial \beta} + \begin{Bmatrix} 2 \\ 22 \end{Bmatrix} \frac{1}{\sqrt{g_{22}}} \cdot \frac{1}{\sqrt{g_{22}}} \right] \right)$$

The second component is

$$-\frac{1}{2g_{22}^{2}} \frac{\partial g_{22}}{\partial \beta} + \frac{1}{g_{22}} \begin{Bmatrix} 2 \\ 22 \end{Bmatrix}$$

But $$\frac{\partial g^{22}}{\partial \beta} = -\frac{1}{g_{22}^{2}} \frac{\partial g_{22}}{\partial \beta} = -2g^{12} \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} - 2g^{22} \begin{Bmatrix} 2 \\ 22 \end{Bmatrix}$$

$$\left(\text{since} \quad \frac{\partial g^{nk}}{\partial x^{l}} = -g^{ni} \begin{Bmatrix} k \\ il \end{Bmatrix} - g^{ki} \begin{Bmatrix} n \\ il \end{Bmatrix} \right)$$

$$= -\frac{2}{g_{22}} \begin{Bmatrix} 2 \\ 22 \end{Bmatrix}$$

$$\therefore \quad -\frac{1}{2g_{22}^{2}} \frac{\partial g_{22}}{\partial \beta} + \frac{1}{g_{22}} \begin{Bmatrix} 2 \\ 22 \end{Bmatrix} = 0$$

$$\therefore \quad \frac{\delta t^{\beta}}{\delta s} = \left(\frac{1}{B^2} \left[-\frac{B}{A^2} \frac{\partial B}{\partial \alpha} \right], \ 0 \right) \equiv \sigma_{(\beta)} \left(-\frac{1}{A}, 0 \right)$$

by 2.2

$$\therefore \quad \sigma_{(\beta)} = \frac{1}{AB} \frac{\partial B}{\partial \alpha} \Rightarrow \frac{\partial \psi_{(\beta)}}{\partial \beta} = \frac{1}{A} \frac{\partial B}{\partial \alpha}$$

Similarly $$\frac{\partial \psi_{(\alpha)}}{\partial \alpha} = -\frac{1}{B} \frac{\partial A}{\partial \beta}$$

Substituting in equations 2.1 gives:

$$\frac{\partial^2 \psi_{(\beta)}}{\partial\alpha\partial\beta} = \frac{\partial^2 \psi_{(\alpha)}}{\partial\alpha\partial\beta} = 0 \qquad (2.3)$$

which are the same as Michell's equations.

3. General Equilibrium Equations

These may be developed by considering the tensor form of the equations of equilibrium of stress i.e.

$$F^j + E,^{ij}{}_i = 0$$

where E_{ij} is the stress tensor and F^j the body force vector, in this case the null vector.

$$E,^{ij}{}_k = \frac{\partial E^{ij}}{\partial x^k} + \begin{Bmatrix} i \\ lk \end{Bmatrix} E^{lj} + \begin{Bmatrix} j \\ lk \end{Bmatrix} E^{il}$$

Also $\quad E^{ij} = g^{ir} g^{js} E_{rs}$

$$\therefore \quad E^{11} = \frac{1}{A^2}\frac{1}{A^2} E_{11} = \frac{E_{11}}{A^4}$$

$$E^{12} = \frac{1}{A^2}\frac{1}{B^2} E_{12} = \frac{E_{12}}{A^2B^2}$$

$$E^{21} = \frac{1}{B^2}\frac{1}{A^2} E_{21} = \frac{E_{21}}{A^2B^2}$$

$$E^{22} = \frac{1}{B^2}\frac{1}{B^2}E_{22} = \frac{E_{22}}{B^4}$$

$$\therefore \quad E^{11}_{,1} = \frac{\partial}{\partial\alpha}\left(\frac{E_{11}}{A^4}\right) + \begin{Bmatrix}1\\11\end{Bmatrix}\frac{E_{11}}{A^4} + \begin{Bmatrix}1\\21\end{Bmatrix}\frac{E_{21}}{A^2B^2}$$

$$+ \begin{Bmatrix}1\\11\end{Bmatrix}\frac{E_{11}}{A^4} + \begin{Bmatrix}1\\21\end{Bmatrix}\frac{E_{12}}{A^2B^2} \qquad (3.1)$$

$$E^{21}_{,2} = \frac{\partial}{\partial\beta}\left(\frac{E_{21}}{A^2B^2}\right) + \begin{Bmatrix}2\\12\end{Bmatrix}\frac{E_{11}}{A^4} + \begin{Bmatrix}2\\22\end{Bmatrix}\frac{E_{21}}{A^2B^2}$$

$$+ \begin{Bmatrix}1\\12\end{Bmatrix}\frac{E_{21}}{A^2B^2} + \begin{Bmatrix}1\\22\end{Bmatrix}\frac{E_{22}}{B^4}$$

As with the strain tensor, the components of the stress tensor do not necessarily have the dimensions of stress.

The physical components of stress are derived in an exactly similar way to the physical components of strain. The stress associated with a plain whose unit normal is the vector l^i, and with a direction specified by the unit vector m^j is $E_{ij}l^im^j$, and to obtain the coordinate stresses the vectors l^i and m^j become unit vectors along the coordinate curves.

$$\text{Unit vector in } \alpha\text{-direction} = \frac{\delta_1^{\ k}}{\sqrt{g_{11}}}$$

$$\text{Unit vector in } \beta\text{-direction} = \frac{\delta_2^{\ 1}}{\sqrt{g_{22}}}$$

$$\therefore \qquad \sigma_{\alpha\alpha} = \frac{E_{11}}{\sqrt{g_{11}}\cdot\sqrt{g_{11}}} = \frac{E_{11}}{A^2}$$

$$\sigma_{\beta\beta} = \frac{E_{22}}{\sqrt{g_{22}}\cdot\sqrt{g_{22}}} = \frac{E_{22}}{B^2}$$

$$\tau_{\alpha\beta} = \tau_{\beta\alpha} = \frac{E_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}} = \frac{E_{21}}{\sqrt{g_{11}}\sqrt{g_{22}}} = \frac{E_{12}}{AB} = \frac{E_{21}}{AB}$$

In this particular system, there are no shear stresses:

$$\therefore \qquad E_{11} = A^2\sigma_{\alpha\alpha}, \quad E_{22} = B^2\sigma_{\beta\beta}, \quad E_{21} = E_{12} = 0$$

From 3.1,

$$E^{11}{}_{,1} + E^{21}{}_{,2} = \frac{\partial}{\partial\alpha}\left(\frac{\sigma_{\alpha\alpha}}{A^2}\right) + 2\left\{\begin{matrix}1\\11\end{matrix}\right\}\frac{\sigma_{\alpha\alpha}}{A^2} + \left\{\begin{matrix}2\\12\end{matrix}\right\}\frac{\sigma_{\alpha\alpha}}{A^2}$$

$$+ \left\{\begin{matrix}1\\22\end{matrix}\right\}\frac{\sigma_{\beta\beta}}{B^2} = 0$$

$$\Rightarrow \qquad \frac{1}{A^2}\frac{\partial\sigma_{\alpha\alpha}}{\partial\alpha} - \frac{2\sigma_{\alpha\alpha}}{A^3}\frac{\partial A}{\partial\alpha} + \frac{2\sigma_{\alpha\alpha}}{A^3}\frac{\partial A}{\partial\alpha} + \frac{1}{A^2 B}\frac{\partial B}{\partial\alpha}\sigma_{\alpha\alpha}$$

$$- \frac{1}{A^2 B}\frac{\partial B}{\partial\alpha}\sigma_{\beta\beta} = 0$$

$$\Rightarrow \quad \frac{1}{A^2}\frac{\partial\sigma_{\alpha\alpha}}{\partial\alpha} + \frac{1}{A^2 B}\frac{\partial B}{\partial\alpha}(\sigma_{\alpha\alpha} - \sigma_{\beta\beta}) = 0$$

$$\Rightarrow \quad \frac{\partial\sigma_{\alpha\alpha}}{\partial\alpha} + \frac{1}{B}\frac{\partial B}{\partial\alpha}(\sigma_{\alpha\alpha} - \sigma_{\beta\beta}) = 0 \qquad (3.2)$$

The second equilibrium equation is derived in a similar fashion

$$E^{12}_{,1} = \frac{\partial}{\partial\alpha}\left(\frac{E_{12}}{A^2B^2}\right) + \left\{\begin{matrix}1\\11\end{matrix}\right\}\frac{E_{12}}{A^2B^2} + \left\{\begin{matrix}1\\21\end{matrix}\right\}\frac{E_{22}}{B^4} + \left\{\begin{matrix}2\\11\end{matrix}\right\}\frac{E_{11}}{A^4}$$

$$+ \left\{\begin{matrix}2\\21\end{matrix}\right\}\frac{E_{12}}{A^2B^2}$$

$$E^{22}_{,2} = \frac{\partial}{\partial\beta}\left(\frac{E_{22}}{B^4}\right) + \left\{\begin{matrix}2\\12\end{matrix}\right\}\frac{E_{12}}{A^2B^2} + \left\{\begin{matrix}2\\22\end{matrix}\right\}\frac{E_{22}}{B^4} + \left\{\begin{matrix}2\\12\end{matrix}\right\}\frac{E_{21}}{A^2B^2}$$

$$+ \left\{\begin{matrix}2\\22\end{matrix}\right\}\frac{E_{22}}{B^4}$$

$$E^{12}_{,1} + E^{22}_{,2} = \left\{\begin{matrix}1\\21\end{matrix}\right\}\frac{\sigma_{\beta\beta}}{B^2} + \left\{\begin{matrix}2\\11\end{matrix}\right\}\frac{\sigma_{\alpha\alpha}}{A^2} + \frac{\partial}{\partial\beta}\left(\frac{\sigma_{\beta\beta}}{B^2}\right) + 2\left\{\begin{matrix}2\\22\end{matrix}\right\}\frac{\sigma_{\beta\beta}}{B^2} = 0$$

$$\Rightarrow \quad \frac{\partial\sigma_{\beta\beta}}{\partial\beta} + \frac{1}{A}\frac{\partial A}{\partial\beta}(\sigma_{\beta\beta} - \sigma_{\alpha\alpha}) = 0 \qquad (3.3)$$

The equations 3.2 and 3.3 apply to a 2-D system of constant thickness, with varying stresses. In this application two thicknesses are defined, both of which may vary, while the stresses remain constant. To obtain the correct equilibrium equations the quantities T_α^{*} and T_β are substituted for $\sigma_{\alpha\alpha}$ and $\sigma_{\beta\beta}$ where :

$$T_\alpha = t_{(\alpha)}\sigma_{\alpha\alpha}$$

$$T_\beta = t_{(\beta)}\sigma_{\beta\beta}$$

$t_{(\alpha)}$ and $t_{(\beta)}$ being the thicknesses in the α- and β-directions respectively. The equilibrium equations are obtained finally in the form :

$$\frac{\partial}{\partial\alpha}(BT_\alpha) - \frac{\partial B}{\partial\alpha}T_\beta = 0$$

$$\frac{\partial}{\partial\beta}(AT_\beta) - \frac{\partial A}{\partial\beta}T_\alpha = 0 \qquad (3.4)$$

The net force between the curves β and $\beta + d\beta$ is $BT_\alpha\, d\beta$ and the net force between the curves α and $\alpha + d\alpha$ is $AT_\beta\, d\alpha$. It is evident from equations 3.4 that

$$\frac{\partial}{\partial\alpha}(BT_\alpha d\beta) \neq 0, \quad \frac{\partial}{\partial\beta}(AT_\beta d\alpha) \neq 0$$

---

*These are second order tensor components, but the double subscript is dropped for convenience in writing. This also brings the notation into line with that used by Hemp (ref. 3).

except for a cartesian system, in which

$$\frac{\partial B}{\partial \alpha} = \frac{\partial B}{\partial \beta} = \frac{\partial A}{\partial \alpha} = \frac{\partial A}{\partial \beta} = 0.$$

This means that the force in a member is not in general constant along its length, and yet the members intersect orthogonally. However, in a framework the condition of orthogonality is never exactly satisfied (although it may, in theory, be approximated as closely as desired by increasing the number of members) and as a result the loads in members along one set of curves vary according to the curvature of the orthogonally intersecting set of curves and the loads in members along these curves.

4. The Solution of the Equilibrium Equations for Michell's Cantilever Equiangular Spirals

For orthogonally intersecting 45° spirals, the equations in polar coordinates are:

$$\begin{aligned} r &= \alpha e^{\theta} \\ r &= \beta e^{-\theta} \end{aligned} \qquad (4.1)$$

$$\therefore \quad r = (\alpha\beta)^{1/2}, \quad \theta = \frac{1}{2}\log\left(\frac{\beta}{\alpha}\right)$$

$$\begin{aligned} \frac{\partial r}{\partial \alpha} &= \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^{1/2}, \quad \frac{\partial r}{\partial \beta} = \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{1/2} \\ \frac{\partial \theta}{\partial \alpha} &= -\frac{1}{2\alpha} \quad, \quad \frac{\partial \theta}{\partial \beta} = \frac{1}{2\beta} \end{aligned} \qquad (4.2)$$

$$ds^2 = dr^2 + r^2 d\theta^2$$

$$dr = \left(\frac{\partial r}{\partial \alpha}\right) d\alpha + \left(\frac{\partial r}{\partial \beta}\right) d\beta = \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}} d\alpha + \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} d\beta$$

$$r\,d\theta = \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\left[\left(\frac{\partial \theta}{\partial \alpha}\right) d\alpha + \left(\frac{\partial \theta}{\partial \beta}\right) d\beta\right] = -\frac{1}{2}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}} d\alpha + \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} d\beta$$

$$\therefore ds^2 = \frac{1}{2}\left(\frac{\beta}{\alpha}\right) d\alpha^2 + \frac{1}{2}\left(\frac{\alpha}{\beta}\right) d\beta^2$$

$$\therefore A^2 = \frac{1}{2}\frac{\beta}{\alpha}, \quad B^2 = \frac{1}{2}\frac{\alpha}{\beta} \qquad (4.3)$$

$$\therefore \frac{\partial A}{\partial \beta} = \frac{1}{2\sqrt{2}}(\alpha\beta)^{-\frac{1}{2}}, \quad \frac{\partial B}{\partial \alpha} = \frac{1}{2\sqrt{2}}(\alpha\beta)^{-\frac{1}{2}}$$

Substituting in equations 3.4 :

$$\frac{1}{\sqrt{2}}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\frac{\partial T_\alpha}{\partial \alpha} + (T_\alpha - T_\beta)\frac{1}{2\sqrt{2}}(\alpha\beta)^{-\frac{1}{2}} = 0$$

$$\frac{1}{\sqrt{2}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}\frac{\partial T_\beta}{\partial \beta} + (T_\beta - T_\alpha)\frac{1}{2\sqrt{2}}(\alpha\beta)^{-\frac{1}{2}} = 0$$

$$\Rightarrow \quad \frac{\partial T_\alpha}{\partial \alpha} + \frac{1}{2\alpha}(T_\alpha - T_\beta) = 0 \qquad (4.4)$$

$$\frac{\partial T_\beta}{\partial \beta} + \frac{1}{2\beta}(T_\beta - T_\alpha) = 0$$

Boundary Conditions

See figure 2.

The force in the edge members must be constant and equal to $F/\sqrt{2}$,

$$\therefore \qquad \frac{F}{\sqrt{2}} \frac{\partial\psi}{\partial\beta}(\beta)\,d\beta = -T_\alpha B d\beta$$

But $\quad \dfrac{\partial\psi}{\partial\beta}(\beta) = \dfrac{1}{A}\dfrac{\partial B}{\partial\alpha}$

$\therefore \qquad$ on tension edge member, $r = \alpha e^{\theta}$,

$$\frac{F}{\sqrt{2}} \frac{1}{A} \frac{\partial B}{\partial\alpha} = -T_\alpha B$$

$$T_\alpha = -\frac{F}{\sqrt{2}} \frac{1}{AB} \frac{\partial B}{\partial\alpha} = -\frac{F}{2\alpha^{1/2}\beta^{1/2}} = -\frac{F}{2r}$$

Similarly on $r = \alpha e^{-\theta}$, $\quad T_\beta = \dfrac{F}{2r}$

There is also a symmetry condition that $T_\alpha(\alpha,\beta) = -T_\beta(\beta,\alpha)$ i.e. interchanging α and β in the expression for either T_α or T_β gives minus the other.

Definition

$$\bar{T}_\alpha = rT_\alpha = \alpha^{1/2}\beta^{1/2}T_\alpha$$

$$\bar{T}_\beta = rT_\beta = \alpha^{1/2}\beta^{1/2}T_\beta$$

Substituting in equations 4.4:

$$\alpha^{-\frac{1}{2}}\beta^{-\frac{1}{2}}\frac{\partial \overline{T}_\alpha}{\partial \alpha} - \frac{1}{2}\alpha^{-\frac{3}{2}}\beta^{-\frac{1}{2}}\overline{T}_\alpha + \frac{1}{2}\alpha^{-\frac{3}{2}}\beta^{-\frac{1}{2}}\overline{T}_\alpha - \frac{1}{2}\alpha\beta^{-\frac{3}{2}}\beta^{-\frac{1}{2}}\overline{T}_\beta = 0$$

$$\alpha^{-\frac{1}{2}}\beta^{-\frac{1}{2}}\frac{\partial \overline{T}_\beta}{\partial \beta} - \frac{1}{2}\beta^{-\frac{3}{2}}\alpha^{-\frac{1}{2}}\overline{T}_\beta + \frac{1}{2}\beta^{-\frac{3}{2}}\alpha^{-\frac{1}{2}}\overline{T}_\beta - \frac{1}{2}\beta^{-\frac{3}{2}}\alpha^{-\frac{1}{2}}\overline{T}_\alpha = 0$$

$$\Rightarrow \quad \frac{\partial \overline{T}_\alpha}{\partial \alpha} = \frac{1}{2\alpha}\overline{T}_\beta$$

$$\frac{\partial \overline{T}_\beta}{\partial \beta} = \frac{1}{2\beta}\overline{T}_\alpha$$

and the boundary conditions become:

$$\alpha = a, \ \overline{T}_\alpha = -\frac{F}{2} \quad : \quad \beta = a, \ \overline{T}_\beta = \frac{F}{2}$$

Now change the independent variables as follows:

$$\alpha = e^{2s'}, \quad \beta = e^{2t'}$$

$$\frac{\partial}{\partial \alpha} = \frac{1}{2\alpha}\frac{\partial}{\partial s'} \ ; \quad \frac{\partial}{\partial \beta} = \frac{1}{2\beta}\frac{\partial}{\partial t'}$$

$\therefore$ The problem becomes:

$$\frac{\partial \overline{T}_\alpha}{\partial s'} = \overline{T}_\beta \qquad s' = \tfrac{1}{2}\log a, \ \overline{T}_\alpha = -\frac{F}{2}$$

$$\frac{\partial \overline{T}_\beta}{\partial t'} = \overline{T}_\alpha \qquad t' = \tfrac{1}{2}\log a, \ \overline{T}_\beta = \frac{F}{2}$$

Finally, let

$$s = \tfrac{1}{2}\log a - s' \qquad (s, t \geqslant 0)$$
$$t = \tfrac{1}{2}\log a - t'$$

and so:

$$\frac{\partial \bar{T}_\alpha}{\partial s} = -\bar{T}_\beta \quad : \quad \frac{\partial \bar{T}_\beta}{\partial t} = -\bar{T}_\alpha \qquad (4\cdot 5)$$
$$\bar{T}_\alpha(0, t) = -\frac{F}{2}, \quad \bar{T}_\beta(s, 0) = \frac{F}{2}$$

or, writing $\bar{T}_\alpha$ as T for simplicity:

$$\frac{\partial^2 T}{\partial s \, \partial t} = T \quad : \quad T(0, t) = -\frac{F}{2} \quad : \quad \left(\frac{\partial T}{\partial s}\right)_{(s, 0)} = -\frac{F}{2}$$

This may be solved by the use of integral transforms, and in particular the Laplace Transform, defined by:

$$\mathcal{L}\left[u(s, t) : t \to p\right] = \int_0^\infty u(s, t) e^{-pt} \, dt = \bar{u}(s, p)$$

$$\therefore \frac{\partial}{\partial s}(\bar{u}) = \overline{\left(\frac{\partial u}{\partial s}\right)} \qquad (4\cdot 6)$$

Also $\mathcal{L}\left[\frac{\partial u}{\partial t} : t \to p\right] = \int_0^\infty \frac{\partial u}{\partial t}(s, t) e^{-pt} \, dt$

$$= \left[u(s,t)\, e^{-pt} \right]_0^\infty - \int_0^\infty u(s,t) \cdot (-p e^{-pt})\, dt$$

$$\Rightarrow \quad \mathcal{L}\left[\frac{\partial u}{\partial t} : t \to p \right] = p\, \mathcal{L}\left[u(s,t) : t \to p \right] - u(s,0)$$

$$(4.7)$$

$\therefore$ Using 4.6 and 4.7:

$$\bar{T} = \overline{\left(\frac{\partial^2 T}{\partial s \partial t} \right)} = \frac{\partial}{\partial s} \overline{\left(\frac{\partial T}{\partial t} \right)} = \frac{\partial}{\partial s} \left[p\bar{T} - T(s,0) \right]$$

$$\therefore \quad p \frac{\partial \bar{T}}{\partial s} - \bar{T} = \left(\frac{\partial T}{\partial s} \right)_{(s,0)} = -\frac{F}{2} \quad \text{by } 4.5$$

$$\therefore \quad \frac{\partial}{\partial s} \left(\bar{T} e^{-s/p} \right) = -\frac{F}{2p} e^{-s/p} = \frac{\partial}{\partial s} \left[\frac{F}{2} e^{-s/p} + f(p) \right]$$

where f is an arbitrary function.

$$\therefore \quad \bar{T}(s,p) = \frac{F}{2} + f(p)\, e^{s/p} \qquad (4.8)$$

But $\bar{T}(0,p) = \int_0^\infty e^{-pt}\, T(0,t)\, dt$

$$= -\int_0^\infty \frac{F}{2} e^{-pt}\, dt = -\frac{F}{2p} \qquad (4.9)$$

$\therefore$ From 4.8 (with $s = 0$) and 4.9:

$$f(p) = -\frac{F}{2} \left(\frac{p+1}{p} \right)$$

Substituting in 4·8 :

$$\overline{T}(s, p) = -\frac{F}{2}\left(\frac{p+1}{p} e^{s/p} - 1\right)$$

$$\therefore T(s, t) = \mathcal{L}^{-1}\left[-\frac{F}{2}\left(e^{s/p} - 1\right) - \frac{F}{2} \cdot \frac{1}{p} e^{s/p} \right]$$

From reference 2, page 244 : $\mathcal{L}^{-1}\left[e^{a/p} - 1\right] = a^{\frac{1}{2}} t^{-\frac{1}{2}} I_1\left(2a^{\frac{1}{2}} t^{\frac{1}{2}}\right)$

page 245 : $\mathcal{L}^{-1}\left[p^{-\nu-1} e^{a/p}\right] = a^{-\frac{1}{2}\nu} t^{\frac{1}{2}\nu} I_\nu\left(2a^{\frac{1}{2}} t^{\frac{1}{2}}\right)$

Put $\nu = 0$; then $\mathcal{L}^{-1}\left[\frac{1}{p} e^{a/p}\right] = I_0\left(2a^{\frac{1}{2}} t^{\frac{1}{2}}\right)$

$$\therefore T(s,t) = -\frac{F}{2}\left[\left(\frac{s}{t}\right)^{\frac{1}{2}} I_1\left(2s^{\frac{1}{2}} t^{\frac{1}{2}}\right) + I_0\left(2s^{\frac{1}{2}} t^{\frac{1}{2}}\right)\right] \qquad (4{\cdot}10)$$

where I_ν is the modified Bessel Function of order ν, defined by:

$$I_\nu(x) = i^{-\nu} J_\nu(ix) = \sum_{r=0}^{\infty} \frac{\left(x/2\right)^{\nu+2r}}{\Gamma(r+1)\,\Gamma(\nu+r+1)}$$

and, in particular,

$$I_1(x) = \sum_{r=0}^{\infty} \frac{\left(x/2\right)^{1+2r}}{\Gamma(r+1)\,\Gamma(r+2)}$$

$$I_0(x) = \sum_{r=0}^{\infty} \frac{\left(x/2\right)^{2r}}{\left[\Gamma(r+1)\right]^2}$$

The following recurrence relations are easily established:

$$xI_\nu'(x) = \nu I_\nu(x) + I_{\nu+1}(x) \qquad (4.11)$$

$$xI_\nu'(x) = -\nu I_\nu(x) + xI_{\nu-1}(x) \qquad (4.12)$$

From equation 4.10

$$\frac{\partial T(s,t)}{ds} = \frac{F}{2}\left[\frac{1}{x} I_1(x) + I_1'(x) + \left(\frac{t}{s}\right)^{1/2} I_0'(x)\right]$$

$$\text{where } x = 2s^{1/2}\, t^{1/2}$$

The first two terms in the bracket are $I_0(x)$, using equation 4.12 with $\nu = 1$. The second term is $(t/s)^{1/2}\, I_1(x)$, using equation 4.11 with $\nu = 0$,

$$\therefore \quad \frac{\partial T}{\partial s} = \frac{F}{2}\left[\left(\frac{t}{s}\right)^{1/2} I_1\,(2s^{1/2}\, t^{1/2}) + I_0\,(2s^{1/2}\, t^{1/2})\right]$$

$\therefore$ From equations 4.5

$$\bar{T}_\alpha = -\frac{F}{2}\left[\left(\frac{s}{t}\right)^{1/2} I_1\,(2s^{1/2}\, t^{1/2}) + I_0\,(2s^{1/2}\, t^{1/2})\right]$$

$$\bar{T}_\beta = \frac{F}{2}\left[\left(\frac{t}{s}\right)^{1/2} I_1\,(2s^{1/2}\, t^{1/2}) + I_0\,(2s^{1/2}\, t^{1/2})\right]$$

Now

$$s = -s' + \tfrac{1}{2}\log a = -\tfrac{1}{2}\log\alpha + \tfrac{1}{2}\log a = \tfrac{1}{2}\log a/\alpha$$

$$t = -t' + \tfrac{1}{2}\log a = -\tfrac{1}{2}\log\beta + \tfrac{1}{2}\log a = \tfrac{1}{2}\log a/\beta$$

and

$$\bar{T}_\alpha = \alpha^{1/2}\, \beta^{1/2}\, T_\alpha \; : \; \bar{T}_\beta = \alpha^{1/2}\, \beta^{1/2}\, T_\beta$$

$$T_\alpha = -\frac{F}{2}(\alpha\beta)^{-\frac{1}{2}}\left[\left(\frac{\log a/\alpha}{\log a/\beta}\right)^{\frac{1}{2}} I_1\left\{\left(\log\frac{a}{\alpha}\cdot\log\frac{a}{\beta}\right)^{\frac{1}{2}}\right\} + I_0\left\{\left(\log\frac{a}{\alpha}\cdot\log\frac{a}{\beta}\right)^{\frac{1}{2}}\right\}\right]$$

$$T_\beta = \frac{F}{2}(\alpha\beta)^{-\frac{1}{2}}\left[\left(\frac{\log a/\beta}{\log a/\alpha}\right)^{\frac{1}{2}} I_1\left\{\left(\log\frac{a}{\alpha}\cdot\log\frac{a}{\beta}\right)^{\frac{1}{2}}\right\} + I_0\left\{\left(\log\frac{a}{\alpha}\cdot\log\frac{a}{\beta}\right)^{\frac{1}{2}}\right\}\right]$$

(4·13)

As $\beta \to a$, $\log\frac{a}{\beta} \to 0$ and $I_1 \to 0$

But $\lim\limits_{\beta\to a}\left(\frac{\log\frac{a}{\alpha}}{\log\frac{a}{\beta}}\right)^{\frac{1}{2}} I_1\left\{\left(\log\frac{a}{\alpha}\cdot\log\frac{a}{\beta}\right)^{\frac{1}{2}}\right\}$

$$= \lim_{x\to 0}\left(\frac{y}{x}\right)^{\frac{1}{2}} I_1\left(x^{\frac{1}{2}}y^{\frac{1}{2}}\right) \quad \text{where} \quad y = \log\frac{a}{\alpha}, \quad x = \log\frac{a}{\beta}$$

$$= \lim_{x\to 0}\left(\frac{y}{x}\right)^{\frac{1}{2}}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}x^{\frac{1}{2}}y^{\frac{1}{2}}\right)^{1+2r}}{\Gamma(r+1)\Gamma(r+2)}$$

$$= \lim_{x\to 0}\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\cdot\frac{1}{2}x^{\frac{1}{2}}y^{\frac{1}{2}}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{4}xy\right)^{r}}{\Gamma(\frac{r}{2}+1)\Gamma(\frac{r}{2}+2)} = \frac{y}{2} = \frac{1}{2}\log\frac{a}{\alpha}$$

$$\therefore \quad T_\alpha(\alpha, a) = -\frac{F}{2}(\alpha a)^{-\frac{1}{2}}\left[\log\frac{a}{\alpha} + 1\right]$$

$$T_\beta(a, \beta) = \frac{F}{2}(a\beta)^{-\frac{1}{2}}\left[\log\frac{a}{\beta} + 1\right]$$

The results of the above analysis are plotted in figure 3.

The dimensionless parameter $-\frac{aT_\alpha}{F}$ is plotted against angular position (θ): the origin coincides with the origin of the spiral system and the line $\theta = 0$ is the axis of symmetry of the beam as shown in figure 2. The steep lines are lines of constant α or β, and the curves joining the intersections of the first set are curves of constant radius. Of these latter curves, the uppermost (labelled "root circle") denotes the circle for which θ_0, in figure 2, has the value π i.e. the edge members intersect on this circle.

The curves also give values of $+\frac{aT_\beta}{F}$ when θ is replaced by $-\theta$.

5. Distribution of Forces at Root of Cantilever

The boundary curve at the root is $r = r_0$, and the direction cosines of the normal, referred to α,β coordinates are $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Thus the applied forces per unit length at the boundary are given by :

$$\begin{bmatrix} P_\alpha \\ \\ P_\beta \end{bmatrix} = \begin{bmatrix} T_\alpha & 0 \\ \\ 0 & T_\beta \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow \begin{array}{l} P_\alpha = -\frac{1}{\sqrt{2}} T_\alpha \\ \\ P_\beta = -\frac{1}{\sqrt{2}} T_\beta \end{array}$$

$$\therefore \quad P_r = -\frac{1}{2}(T_\alpha + T_\beta)$$

$$P_\theta = \frac{1}{2}(T_\alpha - T_\beta)$$

Torque

$$\text{Applied torque} = \tau = \int_{\log\frac{r_0}{a}}^{\log\frac{a}{r_0}} r_0 P_\theta \cdot r_0 \, d\theta$$

$$= \frac{1}{2}\int_{\log\frac{r_0}{a}}^{\log\frac{a}{r_0}} r_0^2 (T_\alpha - T_\beta)\, d\theta = -\int_{\log\frac{r_0}{a}}^{\log\frac{a}{r_0}} r_0^2 T_\beta \, d\theta \quad \text{since } T_\alpha(\theta) = -T_\beta(-\theta)$$

$$\therefore \quad \tau = -\frac{F}{2}\int_{\log\frac{r_0}{a}}^{\log\frac{a}{r_0}} \frac{1}{\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}}\left[\left(\frac{\log\frac{a}{\beta}}{\log\frac{a}{\alpha}}\right)^{\frac{1}{2}} I_1\left\{\left(\log\frac{a}{\alpha}\cdot\log\frac{a}{\beta}\right)^{\frac{1}{2}}\right\} + I_0\left\{\left(\log\frac{a}{\alpha}\cdot\log\frac{a}{\beta}\right)^{\frac{1}{2}}\right\}\right] r_0^2\, d\theta$$

where $\alpha = r_0 e^{-\theta}$, $\beta = r_0 e^{\theta}$.

Putting $\theta_0 = \log\frac{a}{r_0}$ (see figure 2) gives

$$\tau = -\frac{Fr_0}{2}\int_{-\theta_0}^{\theta_0}\left[\left(\frac{\theta_0-\theta}{\theta_0+\theta}\right)^{\frac{1}{2}} I_1\left\{(\theta_0-\theta)^{\frac{1}{2}}(\theta_0+\theta)^{\frac{1}{2}}\right\} + I_0\left\{(\theta_0-\theta)^{\frac{1}{2}}(\theta_0+\theta)^{\frac{1}{2}}\right\}\right] d\theta$$

But $\left(\dfrac{\theta_0-\theta}{\theta_0+\theta}\right)^{\frac{1}{2}} = \dfrac{\theta_0-\theta}{(\theta_0+\theta)^{\frac{1}{2}}(\theta_0-\theta)^{\frac{1}{2}}}$, and if $\phi = (\theta_0+\theta)^{\frac{1}{2}}(\theta_0-\theta)^{\frac{1}{2}}$,

$$\int_{-\theta_0}^{\theta_0} -\frac{\theta}{\phi} I_1(\phi)\, d\theta = 0$$

and $$\tau = -\frac{Fr_0}{2}\int_{-\theta_0}^{\theta_0}\left[\frac{\theta_0}{\phi} I_1(\phi) + I_0(\phi)\right] d\theta$$

The contribution to the torque from the tip load and the loads applied to the edge members at the root is (see figure 2) $F(a - r_0)$. Therefore, for equilibrium of the applied forces, the following must hold:

$$\frac{r_0}{2} \int_{-\Theta_0}^{\Theta_0} \left[\frac{\Theta_0}{\phi} I_1(\phi) + I_0(\phi) \right] d\Theta = a - r_0$$

The above result has been verified graphically, but the analytical evaluation of the integral has not yet been achieved.

Vertical Component of Force

Taking force, W_1, positive downwards

$$W_1 = \int_{-\Theta_0}^{\Theta_0} (P_\Theta \cos\Theta + P_r \sin\Theta)\, r_0 d\Theta$$

$$= \frac{r_0}{2} \int_{-\Theta_0}^{\Theta_0} \left[(T_\alpha - T_\beta)\cos\Theta - (T_\alpha + T_\beta)\sin\Theta \right] d\Theta$$

$$= - r_0 \int_{-\Theta_0}^{\Theta_0} T_\beta (\cos\Theta + \sin\Theta)\, d\Theta$$

since $$T_\alpha(\Theta)\cos\Theta = -T_\beta(-\Theta)\cos(-\Theta)$$

and $$T_\alpha(\Theta)\sin\Theta = T_\beta(-\Theta)\sin(-\Theta)$$

It is seen from figure 2 that the resultant of the other external forces in the vertical direction is

$$F\left(1 + \frac{2}{\sqrt{2}}\sin(\Theta_0 - \pi/4)\right)$$

$$= F(1 + \sin\Theta_0 - \cos\Theta_0)$$

Thus it must be the case that

$$r_0 \int_{-\Theta_0}^{\Theta_0} T_\beta(\cos\Theta + \sin\Theta)\,d\Theta = F(1 + \sin\Theta_0 - \cos\Theta_0)$$

Again, the truth of this has only been verified by graphical integration.

Horizontal Component of Force /

Horizontal Component of Force

Taking force, W_2, positive in direction of tip of beam

$$W_2 = \int_{-\theta_0}^{\theta_0} (P_\theta \sin\theta - P_r \cos\theta)\, r_0 d\theta$$

$$= -\frac{r_0}{2} \int_{-\theta_0}^{\theta_0} \left[(T_\alpha - T_\beta)\sin\theta - (T_\beta + T_\alpha)\cos\theta \right] d\theta$$

$= 0$, as required.

6. Volume of Beam

Total Volume = Volume of Edge Members + Volume of Web

$$V = \left(\frac{1}{f_{(t)}} + \frac{1}{f_{(c)}}\right)\ell - \frac{1}{f_{(c)}} \left[\int_{r_0}^{ae^{\theta}} r dr \int_{-\theta_0}^{0} T_\alpha(r,\theta) d\theta + \int_{r_0}^{ae^{-\theta}} r dr \int_{0}^{\theta_0} T_\alpha(r,\theta) d\theta \right]$$

+

$$+ \frac{1}{f_{(t)}}\left[\int_{r_0}^{ae^{\Theta}} rdr \int_{-\Theta_0}^{0} T_\beta(r,\Theta)d\Theta + \int_{r_0}^{ae^{-\Theta}} rdr \int_{0}^{\Theta_0} T_\beta(r,\Theta)d\Theta\right] \quad (6.1)$$

where $f_{(t)}$ and $f_{(c)}$ are the limiting values of tensile and compressive stress respectively, and l is the length of the edge members, i.e.

$$l = \int_{r_0}^{a} \left[1 + r^2 \left(\frac{d\Theta}{dr}\right)^2\right]^{1/2} dr = \int_{r_0}^{a} \sqrt{2}\, dr$$

$$(\text{since on } r = ae^{\Theta},\ \frac{d\Theta}{dr} = \frac{1}{r}) = \sqrt{2}\,(a - r_0) \quad (6.2)$$

There is, however, another way of calculating the volume of material required, using the expression *:

$$V = \frac{(f_{(t)} + f_{(c)})}{2f_{(t)}\ f_{(c)}} \cdot \frac{1}{\epsilon} \sum_{i=1}^{n} \underset{\sim}{F}_i \cdot \underset{\sim}{u}_i - \frac{(f_{(t)} - f_{(c)})}{2f_{(t)}\ f_{(c)}} \sum_{i=1}^{n} \underset{\sim}{F}_i \cdot \underset{\sim}{r}_i \quad (6.3)$$

where $\underset{\sim}{F}_i$ is the external force vector at joint i and $\underset{\sim}{u}_i$ the corresponding virtual displacement vector. To calculate $\underset{\sim}{u}$ it is necessary to integrate the strain displacement equations obtained in section 2.

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* Reference 3.

$$\varepsilon_{\alpha\alpha} = \frac{1}{A}\frac{\partial u'}{\partial \alpha} + \frac{v'}{AB}\frac{\partial A}{\partial \beta}$$

With the force at the tip applied downwards as shown in figure 2, the α-direction is that of compression. Thus using equations 4.1 to 4.3 :

$$-\varepsilon = \frac{\sqrt{2}}{e^{\theta}}\left[\frac{\partial u'}{\partial \theta}\left(-\frac{1}{2}\frac{e^{\theta}}{r}\right) + \frac{\partial u'}{\partial r}\cdot\frac{1}{2}\left(e^{2\theta}\right)^{\frac{1}{2}}\right]$$

$$+ 2v'\left[\frac{1}{\sqrt{2}}\,e^{\theta}\left(\frac{1}{2re^{\theta}}\right)\right]$$

$$= -\frac{\sqrt{2}}{2}\cdot\frac{1}{r}\frac{\partial u'}{\partial \theta} + \frac{\sqrt{2}}{2}\frac{\partial u'}{\partial r} + \frac{\sqrt{2}}{2}\frac{v'}{r}$$

$$\varepsilon_{\beta\beta} = \frac{1}{B}\frac{\partial v'}{\partial \beta} + \frac{u'}{AB}\frac{\partial B}{\partial \alpha}$$

$$\Rightarrow \quad \varepsilon = \sqrt{2}\,e^{\theta}\left[\frac{\partial v'}{\partial \theta}\left(\frac{1}{2}\cdot\frac{1}{re^{\theta}}\right) + \frac{\partial v'}{\partial r}\,\frac{1}{2}\left(e^{-2\theta}\right)^{\frac{1}{2}}\right]$$

$$+ 2u'\left[-\frac{1}{\sqrt{2}}\,e^{-\theta}\left(-\frac{e^{\theta}}{2r}\right)\right]$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{r}\frac{\partial v'}{\partial \theta} + \frac{\sqrt{2}}{2}\frac{\partial v'}{\partial r} + \frac{\sqrt{2}}{2}\frac{u'}{r}$$

$$\therefore \quad \frac{1}{r}\frac{\partial u'}{\partial \theta} - \frac{\partial u'}{\partial r} - \frac{v'}{r} = \sqrt{2}\,\varepsilon$$

$$\frac{1}{r}\frac{\partial v'}{\partial \theta} + \frac{\partial v'}{\partial r} + \frac{u'}{r} = \sqrt{2}\,\varepsilon$$

Let (u,v) be the physical displacement vector referred to polar coordinates (r,θ). Then (see figure 4),

$$u' = \frac{\sqrt{2}}{2}(u - v) \quad ; \quad v' = \frac{\sqrt{2}}{2}(u + v)$$

$$\frac{1}{r}\left(\frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial \theta}\right) - \left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial r}\right) - \left(\frac{u}{r} + \frac{v}{r}\right) = 2\epsilon$$

$$\frac{1}{r}\left(\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right) + \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r}\right) - \left(\frac{u}{r} - \frac{v}{r}\right) = 2\epsilon$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 2\epsilon$$

i.e. $$\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial r} + \frac{u}{r} = 0$$

The symmetry of the spiral layout demands a solution independent of θ

$$\therefore \quad u = \frac{k_2}{r}$$

$$v = + r\log(k_1 r)^{2\epsilon}$$

where, for $u = v = 0$ at the origin $k_2 = 0$

$$(\lim_{r \to 0} r\log r = 0)$$

$$\therefore \quad u' = -\frac{\sqrt{2}}{2} r\log(k_1 r)^{2\epsilon}$$

$$v' = +\frac{\sqrt{2}}{2} r\log(k_1 r)^{2\epsilon}$$

As a check it may be shown that $\epsilon_{\alpha\beta} = 0$

$$\epsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v'}{B}\right) + \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u'}{A}\right)\right)$$

$$= \frac{1}{2}\left(\begin{array}{l} \frac{1}{2}e^{-\theta}\,(+\,r\log(k_1r)^{2\epsilon})(-\frac{1}{2}\frac{e^{\theta}}{r}) \\ \qquad + \frac{1}{2}e^{-\theta}\,(+\log(k_1r)^{2\epsilon} + 2\epsilon)(\frac{1}{2}\,[e^{2\theta}]^{\frac{1}{2}}) \\ -\frac{1}{2}e^{\theta}\,(r\log(k_1r)^{2\epsilon})(\frac{1}{2}\frac{e^{-\theta}}{r}) \\ \qquad + \frac{1}{2}e^{\theta}\,(-\log(k_1r)^{2\epsilon} - 2\epsilon)(\frac{1}{2}\,[e^{-2\theta}]^{\frac{1}{2}}) \end{array}\right)$$

$$= 0.$$

Thus $\underset{\sim}{u} = (u,v) = (0,\ 2\epsilon r\log(k_1r))$

The part of v proportional to r (i.e. $2\epsilon r\log k_1$) describes a pure rotation and may be ignored

$$\therefore\ u = (0,\ 2\epsilon r\log r)$$

The tangential forces at the root circle, integrated round the circumference (on which $\underset{\sim}{u}$ is constant) must equal Fa/r_0. The radial forces contribute nothing to $\sum_{i=1}^{n} \underset{\sim}{F}_i \cdot \underset{\sim}{u}_i$

$$\therefore\ \frac{1}{\epsilon}\sum_{i=1}^{n} \underset{\sim}{F}_i \cdot \underset{\sim}{u}_i = 2Fa\log a - 2\,\frac{Fa}{r_0}\cdot r_0\log r_0$$

$$= 2Fa\log\frac{a}{r_0} \qquad (6.4)$$

The contribution to $\sum_{i=1}^{n} \underset{\sim}{F}_i \cdot \underset{\sim}{r}_i$ from the tip load and the loads at the root applied to the edge members is obviously zero. Also :

$$\int_{-\Theta_0}^{\Theta_0} T_r \, r d\Theta = \frac{r_0}{2} \int_{-\Theta_0}^{\Theta_0} (T_\beta + T_\alpha) \, d\Theta = 0$$

$$(r = r_0) \qquad \text{since} \quad T_\beta(\Theta) = -T_\alpha(-\Theta)$$

$$\therefore \quad \sum_{i=1}^{n} \underset{\sim}{F}_i \cdot \underset{\sim}{r}_i = 0 \tag{6.5}$$

Thus from equations 6.3 to 6.5

$$\boxed{V = \left(\frac{1}{f_t} + \frac{1}{f_c}\right) \left(F a \log \frac{a}{r_0}\right)} \tag{6.6}$$

Comparing equations 6.6 and 6.1, and equating coefficients of $\frac{1}{f_t}$ and $\frac{1}{f_c}$ (which may be done since f_t and f_c are arbitrary) :

$$-\left\{ \int_{r_0}^{ae^{\theta}} r\,dr \int_{-\theta_0}^{0} T_\alpha(r,\theta)\,d\theta + \int_{r_0}^{ae^{-\theta}} r\,dr \int_{0}^{\theta_0} T_\alpha(r,\theta)\,d\theta \right\}$$

$$= \left\{ \int_{r_0}^{ae^{\theta}} r\,dr \int_{-\theta_0}^{0} T_\beta(r,\theta)\,d\theta + \int_{r_0}^{ae^{-\theta}} r\,dr \int_{0}^{\theta_0} T_\beta(r,\theta)\,d\theta \right\} = F\left(a \log\frac{a}{r_0} + r_0 - a\right)$$

The first equation is obviously satisfied since $T_\alpha(r,\theta) = -T_\beta(r,-\theta)$ and the limits of θ are symmetrical about $\theta = 0$. The second implies that :

$$\frac{1}{2}\int_{r_0}^{ae^{\theta}} r\,dr \int_{-\theta_0}^{0} \left[\left(\frac{\log\frac{a}{r_0} - \theta}{\log\frac{a}{r_0} + \theta}\right)^{\frac{1}{2}} I_1\left\{\left(\log\frac{a}{r_0} - \theta\right)^{\frac{1}{2}}\left(\log\frac{a}{r_0} + \theta\right)^{\frac{1}{2}}\right\} + I_0\left\{\left(\log\frac{a}{r_0} - \theta\right)^{\frac{1}{2}}\left(\log\frac{a}{r_0} + \theta\right)^{\frac{1}{2}}\right\}\right]d\theta$$

$$+\frac{1}{2}\int_{r_0}^{ae^{-\theta}} r\,dr \int_{0}^{\theta_0} \left[\left(\frac{\log\frac{a}{r_0} - \theta}{\log\frac{a}{r_0} + \theta}\right)^{\frac{1}{2}} I_1\left\{\left(\log\frac{a}{r_0} - \theta\right)^{\frac{1}{2}}\left(\log\frac{a}{r_0} + \theta\right)^{\frac{1}{2}}\right\} + I_0\left\{\left(\log\frac{a}{r_0} - \theta\right)^{\frac{1}{2}}\left(\log\frac{a}{r_0} + \theta\right)^{\frac{1}{2}}\right\}\right]d\theta$$

$$= a(\theta_0 - 1) + r_0$$

or, in terms of $\alpha\beta$ coordinates :-

$$\frac{1}{2}\int_{\frac{r_0^2}{\alpha}}^{a} B\,d\beta \int_{\frac{r_0^2}{a}}^{a} (\alpha\beta)^{-\frac{1}{2}} \left[\left(\frac{\log\frac{a}{\beta}}{\log\frac{a}{\alpha}}\right)^{\frac{1}{2}} I_1\left\{\left(\log\frac{a}{\alpha}\cdot\log\frac{a}{\beta}\right)^{\frac{1}{2}}\right\} + I_0\left\{\left(\log\frac{a}{\alpha}\cdot\log\frac{a}{\beta}\right)^{\frac{1}{2}}\right\}\right] A\,d\alpha$$

$$= a\left(\log\frac{a}{r_0} - 1\right) + r_0$$

As before, due to the difficulty of evaluating the integral analytically, this result has not been verified.

7. Conclusions

Considerable mathematical complexity has arisen in dealing with what is basically a simple layout and a simple loading case, which suggests that analytical solutions of more complicated problems may be well nigh impossible to obtain. A more realistic approach to the problem of optimum layouts, limiting the number of possible framework nodes and members has been developed by Hemp and Chan (reference 5). They have demonstrated that such frameworks are Michell structures in the sense that they satisfy the Michell strain criterion when the only strains considered are those between possible framework nodes, and it seems likely that a framework derived in this way would not differ markedly in appearance or weight from a framework whose members formed a straight line approximation to the curves of the exact solution, given that the number of joints was the same in both cases. In view of this there is no point in attempting numerical solutions of the layout equations as the method of reference 5 is simpler.

There is also the problem of relating the Michell structure to something which it is practical to manufacture and, once this has been done, manufacturing to tolerances which can accommodate the large range of internal loading, shown in figure 3, without adding excessively to the overall weight. (In this particular case, the range of internal loading is a feature of the external load system rather than the optimum layout and any practical fully-stressed design would encounter similar difficulties). It seems possible that once the various approximations involved (a) solving the mathematical problem and (b) manufacturing a practical structure have been applied, the advantage over conventional designs may be lost. However, as other researches have pointed out, a knowledge of the optimum is desirable, if only as a basis on which to assess the merits of other designs.

The theory itself is unsatisfactory in that it deals only with pin jointed frameworks, a very limited class of structure: nor is the framework necessarily the lightest structure. It has been shown by Richards and Chan (reference 6) that certain limit designs of plates may be as light as, or lighter than, corresponding optimum plane frameworks, and in some cases may be elastic designs also. However, the design of such plates is based on the corresponding optimum frameworks, so that the Michell Theorem is still fundamental.

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APPENDIX

Alternative Derivation of Equations of Layout Lines

Although it is not always possible to diagonalise an asymmetric second order tensor by a real transformation, there still exist, in 2-dimensional space, two perpendicular directions referred to which the principal diagonal components are extrema.

Consider the arbitrary tensor A_{ij} and the vector L^i: form the scalar λ defined by:-

$$\lambda = \frac{A_{ij} L^i L^j}{g_{lm} L^l L^m} \tag{A.1}$$

(where $g_{lm} L^l L^m$ = square of magnitude of L^i)

The maxima and minima of λ are given by

$$\frac{\partial\lambda}{\partial L^k} = 0$$

Therefore, using the fact that $\dfrac{\partial L^i}{\partial L^k} = \delta^i_k$,

$$\frac{\partial\lambda}{\partial L^k} = 0 = (A_{ij} L^i \delta^j_k + A_{ij} L^j \delta^i_k) g_{lm} L^l L^m - (g_{lm} L^l \delta^m_k) A_{ij} L^i L^j$$

$$\therefore \qquad (A_{ij} L^i \delta^j_k + A_{ij} L^j \delta^i_k) - (g_{ij} L^i \delta^j_k + g_{ij} L^j \delta^i_k)\,\lambda = 0,$$

changing some of the dummy indices

$$\therefore \quad (A_{ik}L^i + A_{kj}L^j) - (g_{\ k}L^i + g_{kj}L^j)\,\lambda = 0$$

$$\therefore \quad (A_{ik}L^i + A_{ki}L^i) - (g_{ik}L^i + g_{ki}L^i)\,\lambda = 0,$$

with a further change of index

$$\therefore \quad \frac{1}{2}(A_{ik} + A_{ki})L^i - \lambda g_{ik}L^i = 0, \qquad (A.2)$$

since g_{ij} is symmetric.

Thus the maxima and minima of λ are the solutions of the determinantal equation:-

$$\left|\bar{A}_{ij} - \lambda g_{ij}\right| = 0 \qquad (A.3)$$

where $\bar{A}_{ij}$ is the symmetric part of A_{ij}.

The directions along which the extreme occur are defined by the unit vectors

$$\frac{L^i_{\ (K)}}{g_{lm}L^l_{\ (K)}L^m_{\ (K)}} = l^i_{\ (K)},$$

say, obtained from equation A.2. In an N-dimensional space there are N orthogonal vectors, even if equation A.3 has repeated roots.

The significance of this is that the tensor used to prove Michell's Theorem need not be symmetric because the quantity $A_{ij}\, l^i_{(K)}\, l^j_{(K)}$ possesses two extremes in any case: the only difference is that whereas if A_{ij} is symmetric, $A_{ij}\, l^i_{(K)}\, l^j_{(M)} = 0$, if A_{ij} is asymmetric this is not necessarily the case, as will be seen.

The condition that the tensor A_{ij} from the symmetric part of $B_{i,j}$ (the compatibility of strain condition) is more complicated than the condition that A_{ij} simply be of the form $B_{i,j}$. This latter condition may be expressed as:-

$$A_{ij,k} = A_{ik,j} \qquad \text{.... (A.4)}$$

The layout lines must coincide everywhere with the principal directions of the symmetric part of A_{ij}. As before these are used to define an $\alpha\,\beta$ coordinate system for which the fundamental tensor is

$$g_{\alpha\beta} = \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix}$$

From equation A.1, since L^i coincides with one of the coordinate directions:

$$\lambda_{(1)} = \frac{A_{11}}{A^2} \qquad \lambda_{(2)} = \frac{A_{22}}{B^2}$$

It is required that $\lambda_{(1)} = -\lambda_{(2)} = e$, say, a constant

$$\therefore \qquad A_{11} = eA^2, \quad A_{22} = -eB^2$$

The optimisation process does not impose any additional constraints on the values of the off-diagonal components: however it is necessary to examine these components to see what constraints, if any, arise as a result of referring them to the $\alpha - \beta$ coordinate system defined above.

Rewrite equation A.2 as :-

$$\frac{1}{2}(A_{ij} + A_{ji})L^i{}_{(K)} - \lambda g_{ij} L^i{}_{(K)} = 0$$

Take the inner product of this equation with $L^j{}_{(M)}$: now the vectors vectors $L^i{}_{(K)}$ and $L^i{}_{(M)}$ are orthogonal and so:

$$g_{ij} L^i{}_{(K)} L^j{}_{(M)} = 0$$

$$\therefore \qquad A_{ij} L^i{}_{(K)} L^j{}_{(M)} = - A_{ji} L^i{}_{(K)} L^j{}_{(M)} \quad (K \neq M)$$

or, dividing by $\left[g_{lm} L^l{}_{(K)} L^m{}_{(K)}\right]^{\frac{1}{2}} \cdot \left[g_{lm} L^l{}_{(M)} L^m{}_{(M)}\right]^{\frac{1}{2}}$,

$$A_{ij} l^i{}_{(K)} l^j{}_{(M)} = - A_{ji} l^i{}_{(K)} l^j{}_{(M)}$$

Quantities of the type $A_{ij} l^i{}_{(K)} l^j{}_{(M)}$ are simply the off-diagonal physical components of the tensor A_{ij} referred to a set of coordinate curves to which the vectors $l^i_{(K)}$ are everywhere tangent i.e. the $\alpha - \beta$ coordinate system. These physical components differ from the actual tensor components by a factor which is the same for both A_{ij} and A_{ji}. Therefore when the tensor A_{ij} is referred to the coordinate system $\alpha - \beta$:

$$A_{ij} = - A_{ji} \qquad (j \neq i) \tag{A.5}$$

Subject to this limitation the off-diagonal components may take on any values.

Let

$$A_{12} = - A_{21} = f(\alpha,\beta)$$

The tensor A_{ij} is thus, in matrix form:

$$\begin{bmatrix} eA^2 & f(\alpha,\beta) \\ -f(\alpha,\beta) & -eB^2 \end{bmatrix}$$

It is required to examine the conditions under which such a tensor will satisfy equation A.4. There are two independent equations viz:

$$A_{12,1} = A_{11,2} \tag{A.6}$$

$$A_{21,2} = A_{22,1} \tag{A.7}$$

In general,

$$A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - \left\{ \begin{matrix} l \\ ik \end{matrix} \right\} A_{lj} - \left\{ \begin{matrix} l \\ jk \end{matrix} \right\} A_{il}$$

$$\therefore \quad A_{12,1} = \frac{\partial A_{12}}{\partial \alpha} - \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} A_{12} - \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} A_{22} - \left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\} A_{11} - \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} A_{12}$$

$$= \frac{\partial f(\alpha,\beta)}{\partial \alpha} - \frac{1}{A}\frac{\partial A}{\partial \beta}.eA^2 + \frac{A}{B^2}\frac{\partial A}{\partial \beta}(-eB^2) - \frac{1}{A}\frac{\partial A}{\partial \alpha} f(\alpha,\beta) - \frac{1}{B}\frac{\partial B}{\partial \alpha} f(\alpha,\beta)$$

$$A_{11,2} = \frac{\partial A_{11}}{\partial \beta} - \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} A_{11} - \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} A_{21} - \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} A_{11} - \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} A_{12}$$

$$= \frac{\partial}{\partial \beta}(eA^2) - \frac{2}{A}\frac{\partial A}{\partial \beta}.eA^2 - (A_{21} + \underbrace{A_{12}}_{=0})\frac{1}{B}\frac{\partial B}{\partial \alpha}$$

Substituting in A.6 gives

$$2eA\frac{\partial A}{\partial \beta} + \frac{\partial f(\alpha,\beta)}{\partial \alpha} - f(\alpha,\beta)\left[\frac{1}{A}\frac{\partial A}{\partial \alpha} + \frac{1}{B}\frac{\partial B}{\partial \alpha}\right] = 0$$

$$\therefore \quad 2e.\frac{1}{B}\frac{\partial A}{\partial \beta} + \frac{1}{AB}\frac{\partial f(\alpha,\beta)}{\partial \alpha} - \frac{1}{(AB)^2} f(\alpha,\beta)\frac{\partial}{\partial \alpha}(AB) = 0$$

$$\therefore \quad 2e.\frac{1}{B}\frac{\partial A}{\partial \beta} + \frac{\partial}{\partial \alpha}\left(\frac{f(\alpha,\beta)}{AB}\right) = 0 \qquad (A.8)$$

Similarly from equation A.7

$$2e\ \frac{1}{A}\frac{\partial B}{\partial \alpha} + \frac{\partial}{\partial \beta}\left(\frac{f(\alpha,\beta)}{AB}\right) = 0 \qquad (A.9)$$

Differentiate A.8 w.r.t. β and A.9 w.r.t. α to obtain, on subtracting:-

$$\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) - \frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) = 0$$

This equation, combined with the equation (on page 13)

$$R^{l}{}_{jnp} = 0$$

gives equations 2.1.

Substituting from 2.1 in the above equations:-

$$\frac{\partial}{\partial\alpha}\left(\frac{f(\alpha,\beta)}{AB}\right) = \frac{\partial}{\partial\beta}\left(\frac{f(\alpha,\beta)}{AB}\right) = 0$$

$$\therefore \qquad f = kAB, \qquad k \text{ a constant}$$

The tensor A_{ij} is thus $\begin{bmatrix} eA^2 & kAB \\ -kAB & -eB^2 \end{bmatrix}$

with physical components $\begin{bmatrix} e & k \\ -k & -e \end{bmatrix}$

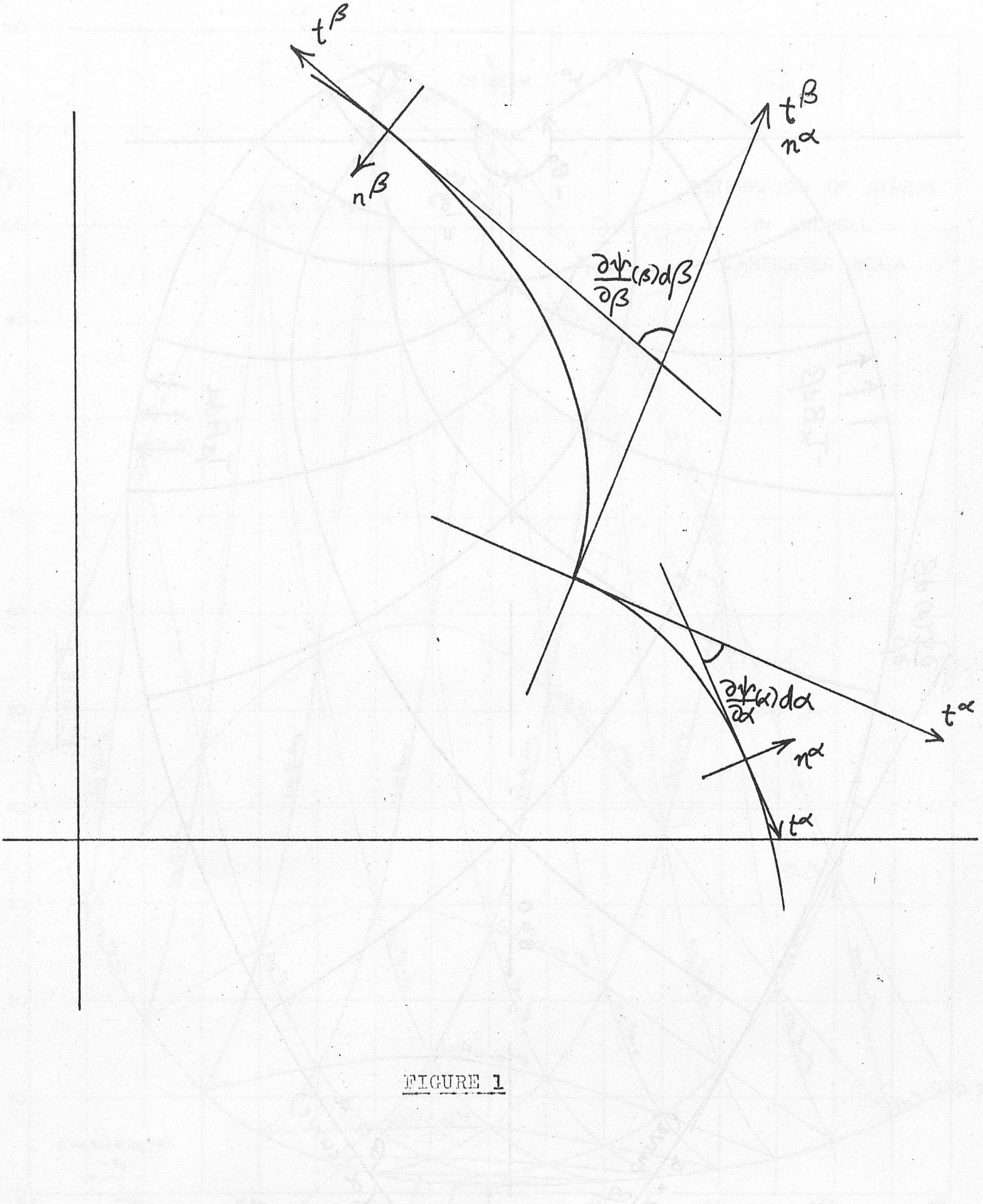
t^β
n^β
t^β
n^α
$\frac{\partial\psi}{\partial\beta}(\beta)d\beta$
$\frac{\partial\psi}{\partial\alpha}(\alpha)d\alpha$
t^α
n^α
t^α

FIGURE 1

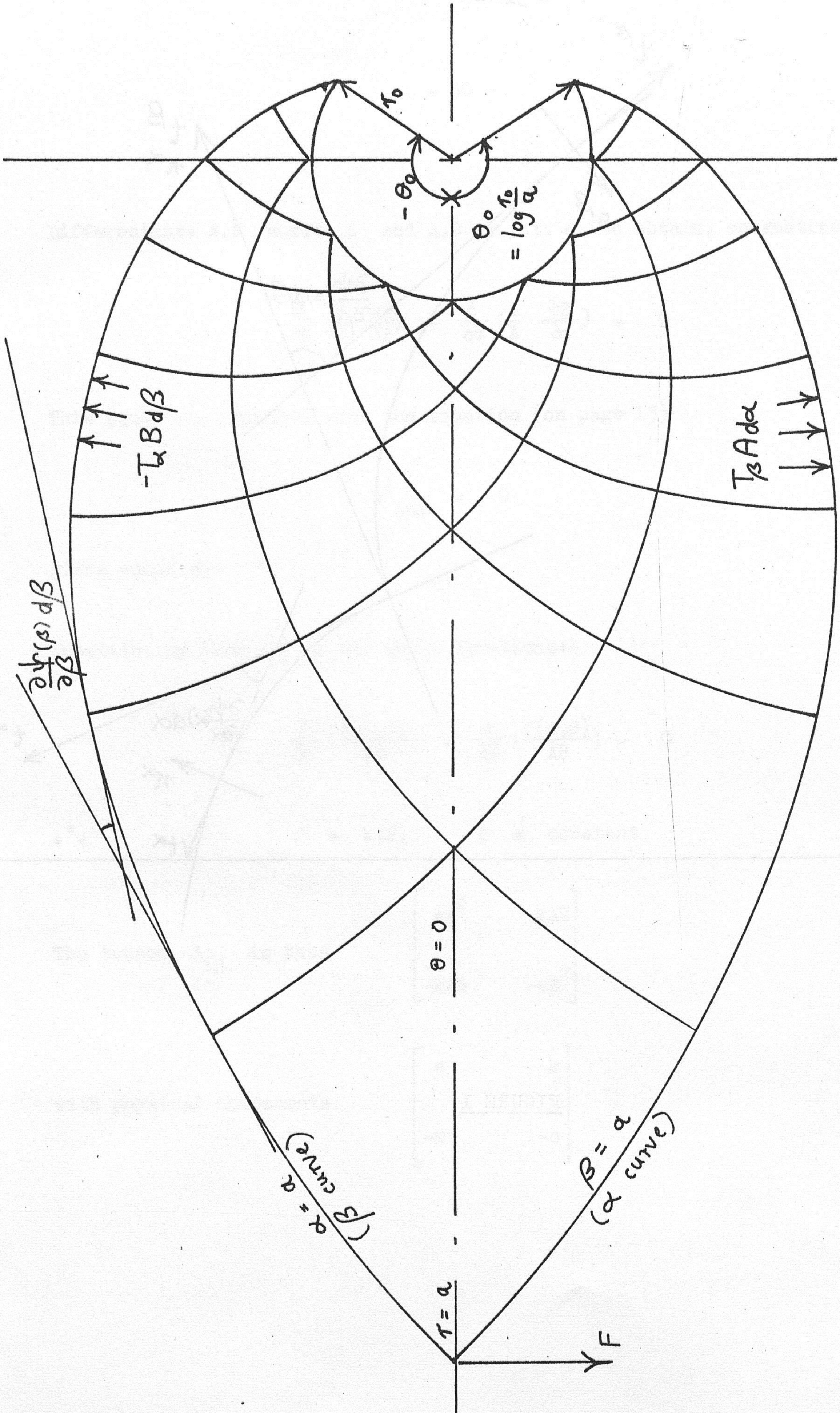
$-T_\alpha B d\beta$
$T_\beta A d\alpha$
$\theta = 0$
$r = a$
F
$\alpha = a$ (β curve)
$\beta = a$ (α curve)
$-\theta_0$
$\theta_0 = \frac{r_0}{\log a}$
r_0

FIGURE 2

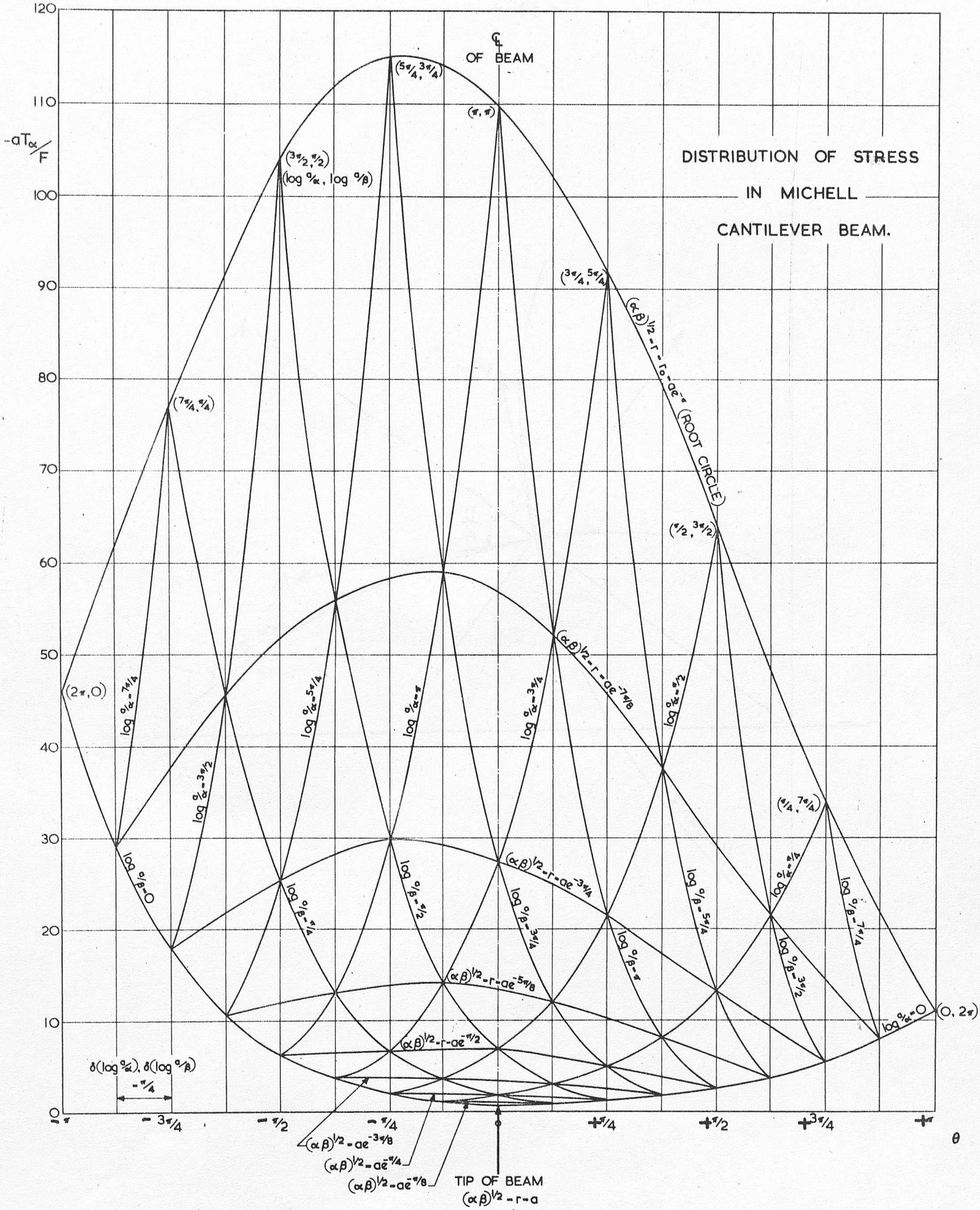
DISTRIBUTION OF STRESS
IN MICHELL
CANTILEVER BEAM.
℄ OF BEAM
TIP OF BEAM
$(\alpha\beta)^{1/2} = r = a$
$(\alpha\beta)^{1/2} = r = r_0 = ae^{\pi}$ (ROOT CIRCLE)
$-aT_\alpha/F$
θ

FIGURE 3

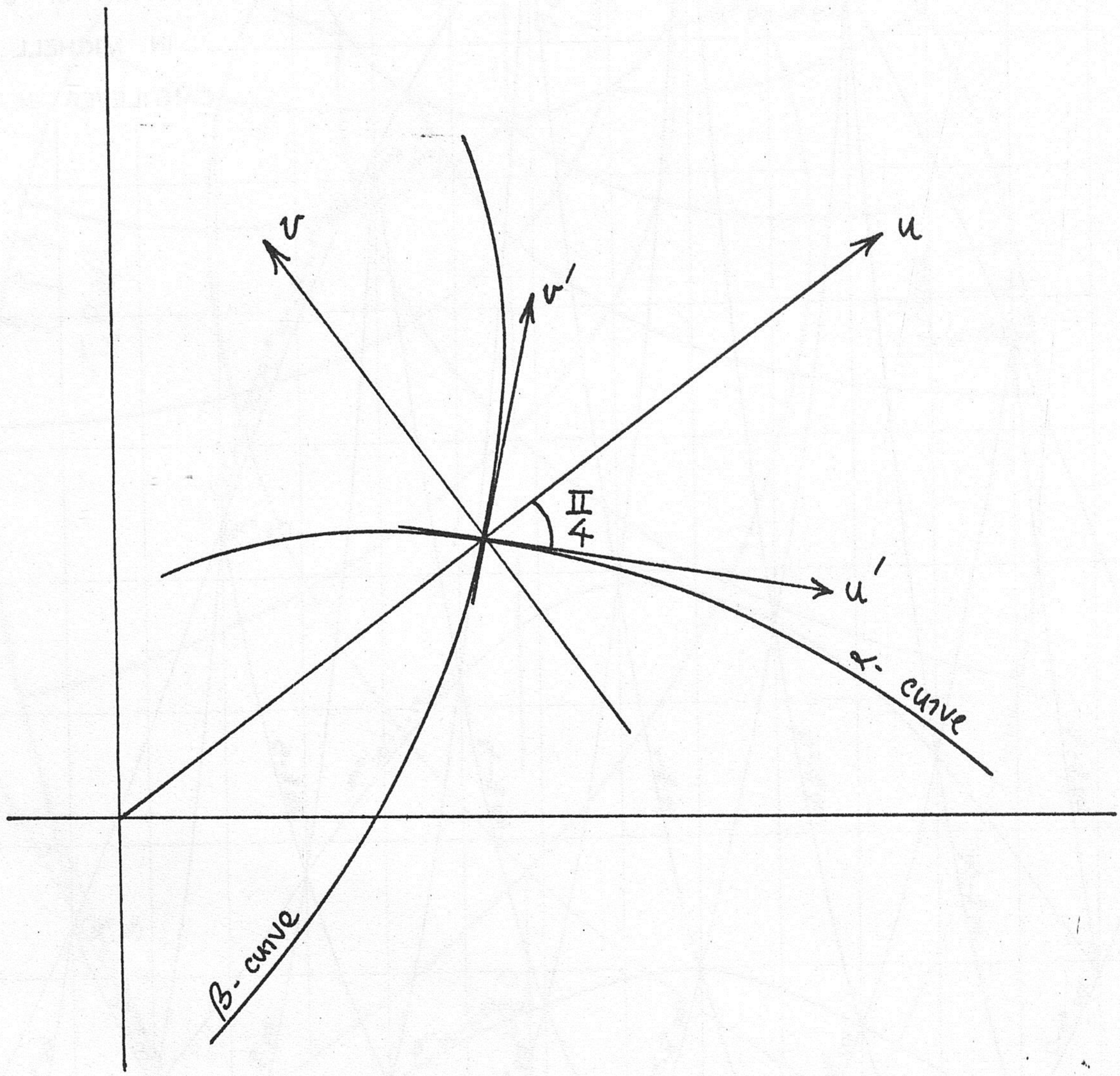

FIGURE 4